On the mean values of some multiplicative functions on the short interval [1]

A.A. Sedunova

Abstract. In this paper we study the mean values of some multiplicative functions connected with the divisor function on the short interval of summation. The asymptotic formulas for such mean values are proved.

1 Introduction

In 1919, S.Ramanujan [1] announced the formula

$$\sum_{n\leqslant X}\frac{1}{\tau(n)}=\frac{X}{\sqrt{\ln X}}\left(A_0+\frac{A_1}{\ln X}+\frac{A_2}{(\ln X)^2}+\ldots+\frac{A_N}{(\ln X)^N}+O\left(\frac{1}{(\ln X)^{N+1}}\right)\right), \qquad (1)$$

where A_j are some constants,

$$A_0=\frac{1}{\sqrt{\pi}}\prod_p\sqrt{p(p-1)}\ \ln\frac{p}{p-1},$$

$\tau(n)$ denotes the number of divisors of n and $N\geqslant 0$ is a fixed integer. The complete proof of (1) was published in 1922 by B.M.Wilson [2]. The general case (with $\tau_k(n)$ instead of $\tau(n)$) was considered by A.Ivić [3] in 1977.

In this paper we generalize (1) and some other theorems of this type to the case when n runs through the short interval of summation, i.e. the interval $x<n\leqslant x+h$, where $x\to+\infty$ and $h\ll x^\alpha$ for a fixed α, $0<\alpha<1$.

Suppose that $k\geqslant 2$ is fixed. The symbols $\sigma(n)$ and $r(n)$ stand for the sum of divisors of n and for the number of representations of n by a sum of two squares $n=x^2+y^2$, respectively. Let us define the multiplicative functions $f_j(n)$, $j=1,2,3$, by the following relations:

$$f_1(n)=\frac{1}{\tau_k(n)};\quad f_2(n)=\frac{\sigma(n)}{\tau(n)};\quad f_3(n)=\begin{cases}\dfrac{1}{r(n)}, & r(n)\neq 0;\\ 0, & r(n)=0.\end{cases}$$

Finally, let

$$S_j(x;h)=\sum_{x<n\leqslant x+h}f_j(n).$$

Our goal is to prove the following theorems.

Theorem 1. *Suppose that $N\geqslant 0$ is a fixed integer. Then the asymptotic formula*

$$S_1(x;h)=\sum_{x<n\leqslant x+h}\frac{1}{\tau_k(n)}=\frac{h}{(\ln x)^{1-\frac{1}{k}}}\left(A_0+\frac{A_1}{\ln x}+\frac{A_2}{(\ln x)^2}+\ldots+\frac{A_N}{(\ln x)^N}+O\left(\frac{1}{(\ln x)^{N+1}}\right)\right),$$

holds for $x\to+\infty$ and $h=x^{\alpha_k}e^{(\ln x)^{0.1}}$, where $\alpha_k=\frac{21k+5}{36k+5}$.

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Here the symbols $A_n = A_n(k)$ denote some positive constants that depend only on n and k. In particular,

$$A_0 = \left(\Gamma\left(\frac{1}{k}\right)\right)^{-1} \prod_p \left(1 - \frac{1}{p}\right)^{\frac{1}{k}} F\left(1, 1, k; \frac{1}{p}\right),$$

where

$$F(a, b, c, z) = 1 + \frac{a \cdot b}{c} z + \frac{a(a+1) \cdot b(b+1)}{c(c+1)} z^2 + \dots$$

is a hypergeometric function.

Corollary 1. *The formula*

$$\sum_{x<n\leqslant x+h} \frac{1}{\tau(n)} = \frac{h}{\sqrt{\ln x}} \left(A_0 + \frac{A_1}{\ln x} + \frac{A_2}{(\ln x)^2} + \dots + \frac{A_N}{(\ln x)^N} + O\left(\frac{1}{(\ln x)^{N+1}}\right)\right),$$

holds true for any fixed $N \geqslant 0$ and for h under the conditions

$$x^{\frac{47}{77}} e^{(\ln x)^{0.1}} \leqslant h \leqslant x.$$

Remark 1. *The below table contains the approximate values of the constants $A_0(k)$ for $2 \leqslant k \leqslant 20$:*

k	$A_0(k)$	k	$A_0(k)$
2	0.54685596	*12*	0.08509329
3	0.35826739	*13*	0.07842606
4	0.26479654	*14*	0.07272702
5	0.20970166	*15*	0.06779967
6	0.17349745	*16*	0.06349728
7	0.14792124	*17*	0.05970809
8	0.12890238	*18*	0.05634549
9	0.11420968	*19*	0.05334130
10	0.10251963	*20*	0.05064114
11	0.09299805		

Theorem 2. *The formula*

$$\sum_{x<n\leqslant x+h} \frac{\sigma(n)}{\tau(n)} = \frac{hx}{\sqrt{\ln x}} \left(B_0 + \frac{B_1}{\ln x} + \frac{B_2}{(\ln x)^2} + \dots + \frac{B_N}{(\ln x)^N} + O\left(\frac{1}{(\ln x)^{N+1}}\right)\right) \quad (2)$$

holds true for any fixed $N \geqslant 0$ and for h under the following conditions:

$$x^{\frac{47}{77}} e^{(\ln x)^{0,1}} \leqslant h \leqslant x.$$

Theorem 3. *The formula*

$$\sum_{x<n\leqslant x+h} \frac{1}{r(n)} = \frac{h}{(\ln x)^{\frac{3}{4}}} \left(C_0 + \frac{C_1}{\ln x} + \frac{C_2}{(\ln x)^2} + \dots + \frac{C_N}{(\ln x)^N} + O\left(\frac{1}{(\ln x)^{N+1}}\right)\right) \quad (3)$$

holds true for any fixed $N \geqslant 0$ and for h under the following conditions:

$$x^{\frac{47}{77}} e^{(\ln x)^{0,1}} \leqslant h \leqslant x.$$

Remark 2. *The coefficients B_n and C_n depend only on n with*

$$B_0 = \frac{1}{2\sqrt{\pi}} \prod_p p\sqrt{\frac{p}{p-1}} \ln\left(1+\frac{1}{p}\right) \approx 0.356903298$$

and

$$C_0 = \frac{2^{\frac{3}{4}}}{\Gamma\left(\frac{1}{4}\right)} \prod_{p\equiv 3 (\mathrm{mod}\, 4)} \left(1-\tfrac{1}{p}\right)^{-\frac{3}{4}} \left(1+\tfrac{1}{p}\right)^{-1} \prod_{p\equiv 1 (\mathrm{mod}\, 4)} \left(1-\tfrac{1}{p}\right)^{\frac{1}{4}} \left(1+\tfrac{1}{2p}+\tfrac{1}{3p^2}+\ldots\right) \approx$$

$$\approx 0.489330926.$$

Remark 3. *The lower bounds for h in the above theorems are not the best possible ones. They can be improved by using more precise upper bounds for $|\zeta(\sigma+it)|$ in the strip $\frac{1}{2} \leqslant \sigma \leqslant 1$.*

Notations

In what follows, $C, C_1, C_2 \ldots$ denote positive absolute constants, which are, generally speaking, different in different relations. The symbol (a, b) stands for the greatest common divisor of integer a and b. Finally, $\theta, \theta_1, \theta_2, \ldots$ denote complex numbers with absolute values not greater than one, which are different in different relations.

2 Auxilliary statements

We need some auxilliary lemmas in order to prove theorems 1 - 3.

Lemma 1. *Let p be a prime number and let $\alpha \geqslant 1$. Then*

$$\tau_k(p^\alpha) = C_{k+\alpha+1}^{k-1},\ \sigma(p^\alpha) = \frac{p^{\alpha+1}-1}{p-1},\ r(p^\alpha) = \begin{cases} 4(\alpha+1),\ if\ p = 4k+1; \\ 4,\ if\ p = 2; \\ 0,\ if\ p = 4k+3\ and\ \alpha\ is\ an\ odd\ number; \\ 4,\ if\ p = 4k+3\ and\ \alpha\ is\ an\ even\ number. \end{cases}$$

Lemma 2. (Perron's formula). *Suppose that the series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely for $\sigma > 1$, $|a_n| \leqslant A(n)$, where $A(n)$ is a positive monotonicially increasing function of n and*

$$\sum_{n=1}^{\infty} |a_n| n^{-\sigma} = O\left((\sigma-1)^{-\alpha}\right)$$

for some $\alpha > 0$, as $\sigma \to 1+0$. Then the formula

$$\sum_{n\leqslant x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s)\frac{x^s}{s} ds + O\left(\frac{x^b}{T(b-1)^\alpha}\right) + O\left(\frac{xA(2x)\ln x}{T}\right)$$

holds true for any b, $1 < b \leqslant b_0$, $T \geqslant 2$, $x = N + \frac{1}{2}$ (the constants in O-symbols depend on b_0).

For a proof of the lemma, see [7], pp. 334-336.

Lemma 3. *The estimate*

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt \ll T\ln T$$

holds true for any $T \geqslant T_0 > 1$.

Proof. This lemma follows immediately from the theorem of Hardy and Littlewood (see, for example, [11], pp. 140-142). ■

Lemma 4. *Let* $\rho(u) = \frac{1}{2} - \{u\}$. *Then the formula*

$$\zeta(s) = \frac{1}{2} + \frac{1}{s-1} + s\int_1^\infty \frac{\rho(u)du}{u^{s+1}},$$

holds true for $s \neq 1$, $\operatorname{Re} s > 0$.

For the proof, see [7], pp. 24-25.

Lemma 5. *The estimates*

$$|\zeta(\sigma+it)| \ll t^{\frac{1-\sigma}{3}}\ln t, \quad |L(\sigma+it,\chi_4)| \ll t^{\frac{1-\sigma}{3}}\ln t$$

hold true for $|t| \geqslant t_0 > 1$ *and* $\frac{1}{2} \leqslant \sigma \leqslant 1 + \frac{1}{\ln t}$.

Proof. This lemma can be easily derived from the approximate equations for $\zeta(s)$ and $L(s,\chi_4)$ (see, for example, [10] and [11], §7, Ch. IV) and from van der Corput's method of estimating of trigonometric sums. ■

Let $N(\sigma,T)$ be the number of zeros of $\zeta(s)$ in the region $\operatorname{Re} s \geqslant \sigma$, $|\operatorname{Im} s| \leqslant T$. Suppose that $q \geqslant 3$ is an integer and let χ be the Dirichlet's character modulo q. Then the symbol $N(\sigma,T;\chi)$ stands for the number of zeros of the function $L(s,\chi)$ in the same domain.

Lemma 6. *The estimates*

$$N(\sigma,T) \ll T^{\frac{12}{5}(1-\sigma)}(\ln T)^{44},$$

$$\sum_{q\leqslant Q}\ \sideset{}{^*}\sum_{\chi \bmod Q} N(\sigma,T;\chi) \ll (Q^2T)^{\frac{12}{5}(1-\sigma)}(\ln QT)^{22}$$

hold uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 1$, $T \geqslant T_0$ *and for* $Q \geqslant 2$ *(the symbol* $\sum^*$ *means the summation over all primitive characters* χ *modulo* q*).*

Lemma 7. *There exist absolute positive constants* t_0 *and* C *such that* $\zeta(s) \neq 0$, $L(s,\chi_4) \neq 0$ *in the region*

$$|t| \geqslant t_0, \quad \sigma \geqslant 1 - \varrho(t), \quad \varrho(t) = C(\ln\ln t)^{-\frac{1}{3}}(\ln t)^{-\frac{2}{3}}.$$

3 Proof of the main results

In this section we give the proofs of theorem 1, 2 and 3.

3.1 The mean-value of the function $\dfrac{1}{\tau_k(n)}$ on the short interval

Suppose that $\sigma = \operatorname{Re} s > 1$ and let

$$F(s) = \sum_{n=1}^{\infty} \frac{1}{\tau_k(n)} \cdot n^{-s}.$$

This series converges absolutely, since

$$|F(s)| \leqslant \sum_{n=1}^{\infty} |a_n| \cdot n^{-\sigma} \leqslant \frac{1}{k} \sum_{n=1}^{\infty} n^{-\sigma} = \frac{1}{k}\left(1 + \int_1^{\infty} \frac{du}{u^{\sigma}}\right) = \frac{1}{k}\left(1 + \frac{1}{\sigma - 1}\right).$$

Setting $a_n = \dfrac{1}{\tau_k(n)}, A(n) \equiv 1, b = 1 + \dfrac{1}{\ln x}, \alpha = 1$ in lemma 2, we get

$$S_1 = S(x, h; f_1) = I + O(R),$$

where

$$I = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{(x+h)^s - x^s}{s} ds, \quad R = \frac{x^b}{T(b-1)} + \frac{xA(2x)\ln x}{T} \ll \frac{x \ln x}{T}.$$

Further, $F(s) = \prod_p F_p(s)$, where

$$F_p(s) = 1 + \frac{1}{\tau_k(p)p^s} + \frac{1}{\tau_k(p^2)p^{2s}} + \ldots = 1 + \frac{1!}{kp^s} + \frac{2!}{k(k+1)p^{2s}} + \ldots$$

Writing $F_p(s)$ in the form

$$F_p(s) = \left(1 - \frac{1}{p^s}\right)^{-\frac{1}{k}} \left(1 - \frac{1}{p^{2s}}\right)^{m_k} G_p(s), \quad \text{where} \quad m_k = \frac{(k-1)^2}{2k^2(k+1)},$$

we obtain

$$F(s) = \frac{(\zeta(s))^{\frac{1}{k}}}{(\zeta(2s))^{m_k}} G(s),$$

where

$$G(s) = \prod_p G_p(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-\frac{1}{k}} \left(1 - \frac{1}{p^{2s}}\right)^{m_k} (1 + u(s) + v(s)),$$

$$u(s) = \frac{1}{kp^s} + \frac{2}{k(k+1)p^{2s}}, \; v(s) = \frac{3!}{k(k+1)(k+2)p^{3s}} + \frac{4!}{k(k+1)(k+2)(k+3)p^{4s}} + \ldots$$

Now we continue the function $F(s)$ to the left of the line $\operatorname{Re} s = 1$. Suppose that $\frac{1}{2} \leqslant \sigma \leqslant 1$. Then the following estimates hold true:

$$|u(s)| \leqslant \frac{1}{kp^{\sigma}} \left(1 + \frac{2}{k+1} \frac{1}{p^{\sigma}}\right) \leqslant \left(1 + \frac{2}{3\sqrt{2}}\right) \frac{1}{kp^{\sigma}} < \frac{3}{2} \frac{1}{kp^{\sigma}};$$

$$|v(s)| \leqslant \frac{1}{3kp^\sigma}\left(\frac{2\cdot 3}{3\cdot 4} + \frac{2\cdot 3\cdot 4}{3\cdot 4\cdot 5}\frac{1}{p^\sigma} + \ldots\right) \leqslant \frac{1}{2kp^{3\sigma}}\left(1 + \frac{1}{p^\sigma} + \ldots\right) \leqslant \frac{7}{4}\frac{1}{kp^\sigma};$$

$$|u(s)+v(s)| \leqslant \frac{3}{2k}\frac{1}{kp^\sigma} + \frac{7}{4k}\frac{1}{p^{3\sigma}} < \frac{5}{2}\frac{1}{kp^\sigma};$$

$$|u(s)\cdot v(s)| \leqslant \frac{3}{2k}\frac{7}{4k}\frac{1}{p^{4\sigma}} = \frac{21}{8k^2}\frac{1}{p^{4\sigma}} \leqslant \frac{21}{16\sqrt{2}}\frac{1}{kp^{3\sigma}} < \frac{1}{kp^{3\sigma}}.$$

Now let us consider the expansion

$$\ln(1+u(s)+v(s)) = (u+v) - \frac{1}{2}(u+v)^2 + \frac{1}{3}(u+v)^3 - \ldots$$

Obviously, we have

$$\left|\frac{1}{3}(u+v)^3 - \frac{1}{4}(u+v)^4 + \ldots\right| \leqslant \left(\frac{5}{2}\frac{1}{kp^\sigma}\right)^3 + \frac{1}{4}\left(\frac{5}{2}\frac{1}{kp^\sigma}\right)^4 + \ldots \leqslant$$

$$\leqslant \frac{1}{3}\left(\frac{5}{2}\frac{1}{kp^\sigma}\right)^3 \frac{1}{1-\frac{5}{2kp^\sigma}} \leqslant \frac{1}{3}\left(\frac{5}{2}\right)^3\frac{1}{2^2}\left(1-\frac{5}{4\sqrt{2}}\right)^{-1}\frac{1}{kp^{3\sigma}} < \frac{45}{4}\frac{1}{kp^{3\sigma}}.$$

Next,

$$(u+v) - \frac{1}{2}(u+v)^2 = \left(u - \frac{u^2}{2}\right) + \left(v - \frac{v^2}{2} - uv\right). \tag{4}$$

The second term on the right hand of (4) does not exceed in absolute value

$$\frac{7}{4k}\frac{1}{p^{3\sigma}} + \frac{1}{2}\left(\frac{7}{4k}\right)^2\frac{1}{p^{6\sigma}} + \frac{1}{kp^{3\sigma}} < \frac{3}{kp^{3\sigma}}.$$

Moreover,

$$u - \frac{u^2}{2} = \frac{1}{kp^s} + \frac{2}{k(k+1)}\frac{1}{p^{2s}} - \frac{1}{2k^2p^{2s}} + \frac{3\theta}{4}\frac{1}{kp^{3\sigma}} = \frac{1}{kp^s} + \frac{3k-1}{2k^2(k+1)}\frac{1}{p^{2s}} + \frac{3\theta}{4}\frac{1}{kp^{3\sigma}}.$$

Therefore,

$$\ln(1+u(s)+v(s)) = \frac{1}{kp^s} + \frac{3k-1}{2k^2(k+1)}\frac{1}{p^{2s}} + \frac{15\theta_1}{kp^{3s}}.$$

Further,

$$\ln\left(1-\frac{1}{p^s}\right) = -\frac{1}{p^s} - \frac{1}{2p^{2s}} - \ldots = -\frac{1}{p^s} - \frac{1}{2p^{2s}} + \frac{7\theta_2}{6}\frac{1}{p^{3\sigma}},$$

$$\ln\left(1-\frac{1}{2p^{2s}}\right) = -\frac{1}{p^{2s}} - \ldots = -\frac{1}{p^{2s}} + \frac{5\theta_3}{4}\frac{1}{p^{3\sigma}}.$$

Hence

$$\ln G_p(s) = \ln(1+u(s)+v(s)) + \frac{1}{k}\ln\left(1-\frac{1}{p^s}\right) - m_k\ln\left(1-\frac{1}{p^{2s}}\right) =$$

$$= \frac{1}{kp^s} + \frac{3k-1}{2k^2(k+1)}\frac{1}{p^{2s}} + \frac{3\theta_1}{4}\frac{19}{kp^{3\sigma}} - \frac{1}{kp^s} - \frac{1}{2kp^{2s}} + \frac{7\theta_2}{6}\frac{1}{kp^{3\sigma}} + \frac{m_k}{p^{2s}} + \frac{5\theta_3}{4}\frac{m_k}{p^{3\sigma}} = \frac{16\theta}{p^{3\sigma}}.$$

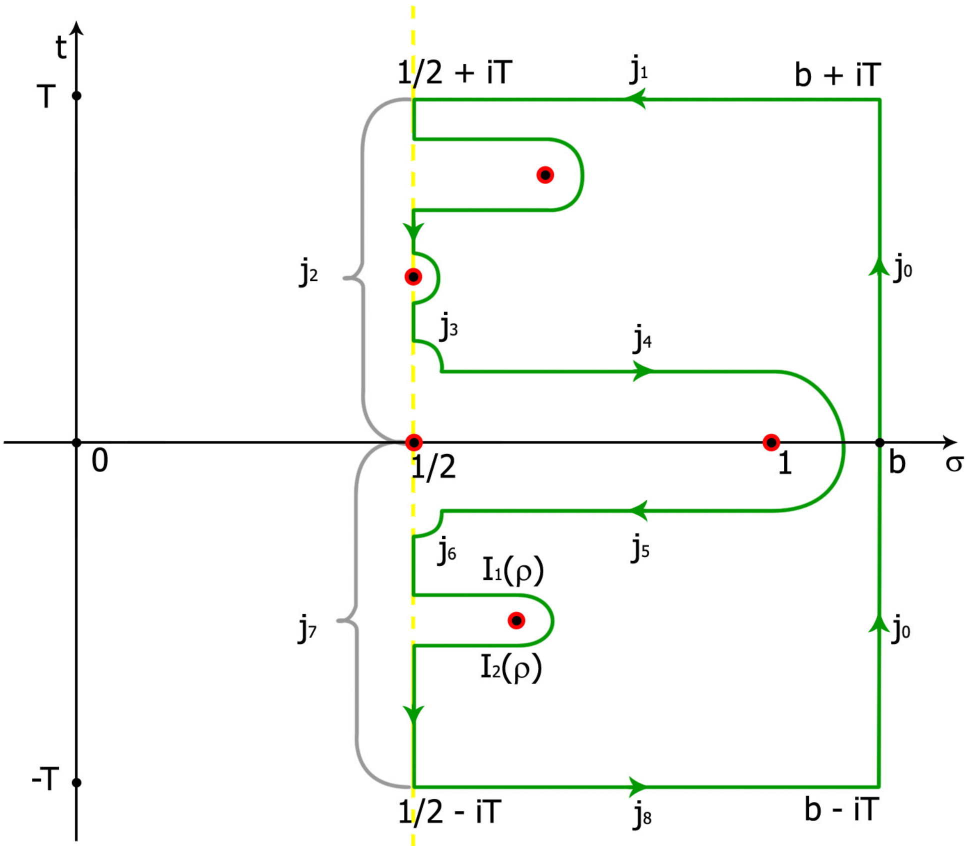
Finally, for $\frac{1}{2} \leqslant \sigma \leqslant 1$ we get

$$\left| \sum_p \ln G_p(s) \right| \leqslant \frac{19}{k} \sum_p \frac{1}{p^{\frac{3}{2}}} < \frac{19}{k} < \frac{19}{2} = 9.5,$$

$$-9.5 \leqslant \ln |G(s)| \leqslant 9.5, \quad e^{-9.5} \leqslant |G(s)| \leqslant e^{9.5}.$$

Let Γ be the boundary of the rectangle with the vertices $\frac{1}{2} \pm iT, b \pm iT$, where the zeros of $\zeta(s)$ of the form $\frac{1}{2} + i\gamma$, $|\gamma| < T$, are avoided by the semicircles of the infinitely small radius lying to the right of the line $\operatorname{Re} s = \frac{1}{2}$, the pole of $\zeta(2s)$ at the point $s = \frac{1}{2}$ is avoided by two arcs Γ_1 and Γ_2 with the radius $\frac{1}{\ln x}$, and let a horizontal cut be drawn from the critical line inside this rectangle to each zero $\rho = \beta + i\gamma$, $\beta > \frac{1}{2}$, $|\gamma| < T$. Then the function $F(s)$ is analytic inside Γ. By the Cauchy residue theorem,

$$j_0 = -\sum_{k=1}^{8} j_k - \sum_{\rho} j_\rho = -(j_4 + j_5) - \sum_{k \neq 4,5} j_k - \sum_{\rho} j_\rho.$$



By lemma 5,

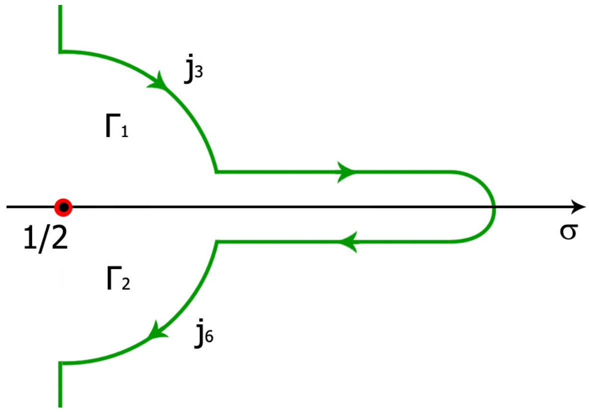
$$F(s) \ll T^{\frac{1-\sigma}{3k}} (\ln T)^{m_k} \ll T^{\frac{1-\sigma}{3k}} (\ln T)^{0.6}.$$

Then

$$|j_1| = \left| \frac{1}{2\pi i} \int_{b+iT}^{\frac{1}{2}+iT} F(s) \frac{(x+h)^s - x^s}{s} ds \right| \ll \frac{1}{T} \int_{\frac{1}{2}}^{b} T^{\frac{1-\sigma}{3k}} \cdot (\ln x)^{0.6} x^{\sigma} d\sigma \ll$$

$$\ll \frac{x}{T} \int_{\frac{1}{2}}^{b} \frac{x^{\sigma-1}}{T^{\frac{\sigma-1}{3k}}} (\ln x)^{0.6} d\sigma \ll \frac{x}{T} \int_{\frac{1}{2}}^{b} \left(\frac{x}{T^{\frac{1}{3k}}} \right)^{\sigma-1} (\ln x)^{0.6} d\sigma \ll \frac{x}{T} (\ln x)^{0.6}.$$

The similar estimate is valid for j_8.



By lemma 4, on Γ_1, Γ_2 we have:

$$|\zeta(s)| = \left| 0.5 + \frac{1}{s-1} + s \int_1^{\infty} \frac{\rho(u)}{u^{s+1}} du \right| \leqslant 0.5 + 2 + 0.1 + 0.6 \cdot 0.5 \int_1^{\infty} \frac{du}{u^{\frac{3}{2}}} \leqslant 3.2,$$

$$|\zeta(2s)| \geqslant \frac{1}{|2s-1|} - 1.1 \geqslant 0.5 \ln x - 1.1 > 0.4 \ln x.$$

Hence

$$|F(s)| \leqslant |G(s)| \frac{3.2^{\frac{1}{k}}}{(0.4 \cdot \ln x)^{m_k}} < 10.$$

Therefore,

$$|j_3 + j_6| \leqslant \frac{1}{2\pi} \int_{\Gamma_1 \cup \Gamma_2} |F(s)| \left| \frac{(x+h)^s - x^s}{s} \right| ds \leqslant \frac{10}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{2 \cdot (2x)^{\frac{1}{2} + \frac{1}{\ln x}}}{\frac{1}{2}} \cdot \frac{d\varphi}{\ln x} \ll \frac{\sqrt{x}}{\ln x}.$$

Further,

$$|F(s)| \leqslant |\zeta(s)|^{\frac{1}{k}} (\ln x)^{m_k} |G(s)| \ll (\ln x)^{m_k} |\zeta(\sigma + it)|^{\frac{1}{k}}.$$

Hence

$$|j_2| = \left| \text{p.v.} \frac{1}{2\pi i} \int_{\frac{1}{2}+iT}^{\frac{1}{2}+\frac{i}{\ln x}} F(s) \cdot \frac{(x+h)^s - x^s}{s} ds \right| \ll \int_{\frac{1}{\ln x}}^{T} (\ln x)^{m_k} \cdot \left| \zeta\left(\tfrac{1}{2} + it\right) \right|^{\frac{1}{k}} \sqrt{x} \frac{dt}{t+1} \ll$$

$$\ll (\ln x)^{m_k} \sqrt{x} \int_0^T \left| \zeta\left(\tfrac{1}{2} + it\right) \right|^{\frac{1}{k}} \frac{dt}{t+1} = (\ln x)^{m_k} \sqrt{x} \sum_{\nu \geqslant 0} \int_{T/2^{\nu}}^{T/2^{\nu+1}} \frac{|\zeta(\frac{1}{2} + it)|^{\frac{1}{k}}}{t+1} dt.$$

Denoting the summands in the last sum by $j(\nu)$ and taking $X = T \cdot 2^{-\nu}$, by the Hölder inequality we get:

$$j(\nu) \ll \frac{1}{X}\left(\int_X^{2X} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt\right)^{\frac{1}{2k}} X^{1-\frac{1}{2k}} \ll \frac{1}{X}\left(X\ln X\right)^{\frac{1}{2k}} X^{1-\frac{1}{2k}} \ll (\ln X)^{\frac{1}{2k}} \ll (\ln T)^{\frac{1}{2k}}.$$

Hence,

$$\sum_{\nu \geqslant 0} j(\nu) \ll (\ln T)^{1+\frac{1}{2k}} \ll (\ln T)^{\frac{5}{4}}.$$

Then the upper bound for j_2 has the form

$$|j_2| \ll (\ln x)^{m_k}\sqrt{x}(\ln x)^{\frac{5}{4}} = \sqrt{x}(\ln x)^{m_k+\frac{5}{4}}.$$

The integral j_7 is estimated as above.

The main term arises from the calculation of j_4 and j_5. Let us define the entire function $w(s)$ by the relation

$$\zeta(s) = \frac{w(s)}{s-1}$$

and let $s = 1-u+i\cdot 0$, where $0 \leqslant u \leqslant \frac{1}{2}$. Then

$$\sqrt[k]{\zeta(s)} = \frac{\sqrt[k]{w(s)}}{\sqrt[k]{-u+i\cdot 0}}.$$

Since $-u+i\varepsilon \to u\cdot e^{\pi i}$ as $\varepsilon \to +0$, then

$$\sqrt[k]{-u+i\cdot 0} = \sqrt[k]{u}e^{\frac{\pi i}{k}}, \quad \sqrt[k]{\zeta(s)} = \frac{\sqrt[k]{w(\sigma)}}{\sqrt[k]{u}}e^{-\frac{\pi i}{k}}.$$

Therefore, on the upper edge of the cut we have

$$F(s) = \frac{\sqrt[k]{w(1-u)}}{(\zeta(2-2u))^{m_k}}\, G(1-u)\frac{e^{-\frac{\pi i}{k}}}{\sqrt[k]{u}} = \frac{\Pi(u)e^{-\frac{\pi i}{k}}}{\sqrt[k]{u}},$$

where

$$\Pi(u) = G(1-u)\frac{\sqrt[k]{w(1-u)}}{(\zeta(2-2u))^{m_k}}.$$

Hence,

$$j_4 = \frac{1}{2\pi i}\int_{\frac{1}{2}+\frac{1}{\ln x}+i\cdot 0}^{1+i\cdot 0} F(\sigma+i\cdot 0)\frac{(x+h)^s - x^s}{s}ds =$$

$$= \frac{1}{2\pi i}\int_{\frac{1}{2}+\frac{1}{\ln x}+i\cdot 0}^{1+i\cdot 0} F(\sigma+i\cdot 0)\int_0^h (x+u)^{s-1}du\, ds =$$

$$= \frac{1}{2\pi i}\int_x^{x+h}\int_{\frac{1}{2}+\frac{1}{\ln x}}^{1} F(\sigma+i\cdot 0)y^{\sigma-1}d\sigma\, dy = \frac{e^{-\frac{\pi i}{k}}}{2\pi i}\int_x^{x+h}\int_0^{\frac{1}{2}-\frac{1}{\ln x}}\frac{\Pi(u)y^{-u}}{\sqrt[k]{u}}du\, dy.$$

Suppose that $N \geqslant 0$ is fixed. Then

$$\Pi(u) = \Pi_0 + \Pi_1 u + \Pi_2 u^2 + \ldots + \Pi_N u^N + O_N(u^{N+1}),$$

where

$$\Pi_0 = \Pi(0) = \frac{\sqrt[k]{w(1)}}{(\zeta(2))^{m_k}}\, G(1) = \frac{\prod_p \left(1-\frac{1}{p}\right)^{\frac{1}{k}} \left(1-\frac{1}{p^2}\right)^{m_k} \left(1+\frac{1}{kp}+\ldots\right)}{(\zeta(2))^{m_k}} =$$

$$= \prod_p \left(1-\frac{1}{p}\right)^{\frac{1}{k}} \left(1+\frac{1}{kp}+\ldots\right).$$

Thus, we have

$$j_4 = \frac{e^{-\frac{\pi i}{k}}}{2\pi i} \int_x^{x+h} \left(\sum_{0 \leqslant n \leqslant N} \Pi_n \int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{u^n y^{-u}}{\sqrt[k]{u}} du + O\left(J\right) \right) dy,$$

where

$$J = \int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{u^{N+1} y^{-u}}{\sqrt[k]{u}} du \leqslant \frac{\Gamma\left(N+2-\frac{1}{k}\right)}{(\ln y)^{N+2-\frac{1}{k}}}.$$

Using the estimate

$$\int_\lambda^\infty w^{k-\gamma} e^{-w} dw < ek!\, \lambda^{k-\gamma} e^{-\lambda},$$

where $\lambda > 1$, $0 < \gamma < 1$, $k \geqslant 1$, we easily get

$$\int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{u^n y^{-u}}{\sqrt[k]{u}} du = \frac{1}{(\ln y)^{n+1-\frac{1}{k}}} \int_0^{\ln y\left(\frac{1}{2}-\frac{1}{\ln x}\right)} e^{-w} w^{n-\frac{1}{k}} dw =$$

$$= \frac{1}{(\ln y)^{n+1-\frac{1}{k}}} \left(\int_0^\infty e^{-w} w^{n-\frac{1}{k}} dw + \frac{\theta e n! (\ln y)^{n-\frac{1}{k}}}{\sqrt{y}} \right) = \frac{\Gamma\left(n+1-\frac{1}{k}\right)}{(\ln y)^{n+1-\frac{1}{k}}} + \frac{\theta e n!}{\sqrt{y}\ln y}.$$

Therefore,

$$j_4 = \frac{e^{-\frac{\pi i}{k}}}{2\pi i} \int_0^h \sum_{0 \leqslant n \leqslant N} \frac{\Pi_n \Gamma(n+1-\frac{1}{k})}{(\ln(x+u))^{n+1-\frac{1}{k}}} du + O\left(\frac{h}{(\ln x)^{N+2-\frac{1}{k}}} \right).$$

Let

$$\varphi(x) = \frac{1}{(\ln x)^{n+1-\frac{1}{k}}}.$$

Then the Lagrange mean-value theorem yields

$$\varphi(x+u) = \varphi(x) + u\varphi'(x+\theta_1 u) = \frac{1}{(\ln x)^{n+1-\frac{1}{k}}} + \frac{\theta_2 h \left(n+1-\frac{1}{k}\right)}{x\,(\ln x)^{n+2-\frac{1}{k}}}.$$

Thus we get

$$j_4 = \frac{h e^{-\frac{\pi i}{k}}}{2\pi i} \sum_{0 \leqslant n \leqslant N} \frac{\Pi_n \Gamma\left(n+1-\frac{1}{k}\right)}{(\ln x)^{n+1-\frac{1}{k}}} + O\left(\frac{h}{(\ln x)^{N+2-\frac{1}{k}}} \right) + O\left(\frac{h^2}{x} \frac{1}{(\ln x)^{N+2-\frac{1}{k}}} \right),$$

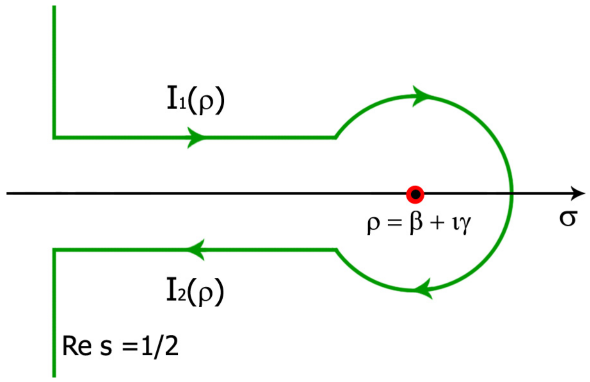
$$j_5 = -\frac{he^{\frac{\pi i}{k}}}{2\pi i} \sum_{0\leqslant n\leqslant N} \frac{\Pi_n \Gamma\left(n+1-\frac{1}{k}\right)}{(\ln x)^{n+1-\frac{1}{k}}} + O\left(\frac{h}{(\ln x)^{N+2-\frac{1}{k}}}\right) + O\left(\frac{h^2}{x}\frac{1}{(\ln x)^{N+2-\frac{1}{k}}}\right).$$

Finally,

$$-(j_4+j_5) = -\frac{h}{(\ln x)^{1-\frac{1}{k}}}\left(\sum_{0\leqslant n\leqslant N} \frac{(-1)^n \Pi_n}{\Gamma\left(\frac{1}{k}-n\right)(\ln x)^n} + O\left(\frac{1}{(\ln x)^{N+1}}\right) + O\left(\frac{h}{x(\ln x)^{N+1}}\right)\right).$$

It remains to estimate the sum

$$\sum_{|\gamma|<T} j_\rho,\ j_\rho = I_1(\rho) + I_2(\rho),\ \text{where}\ \ \rho = \beta + i\gamma.$$



Since

$$\left|\frac{(x+h)^s - x^s}{s}\right| = \left|\int_0^h (x+u)^{s-1}du\right| \ll \int_0^h (x+u)^{\sigma-1}du \ll hx^{\sigma-1},$$

then

$$I_1(\rho) \ll \int_{\frac{1}{2}}^{\beta} (\ln x)^{m_k} |\zeta(\sigma+i\gamma)|^{\frac{1}{k}} hx^{\sigma-1} d\sigma \ll \frac{h}{x}(\ln x)^{m_k}\int_{\frac{1}{2}}^{\beta} x^\sigma |\zeta(\sigma+i\gamma)|^{\frac{1}{k}} d\sigma$$

and the same estimate is valid for $I_2(\rho)$. Hence,

$$|j_\rho| \ll \int_{\frac{1}{2}}^{\beta} hx^{\sigma-1}(\ln x)^{m_k} T^{\frac{1-\sigma}{3k}} (\ln x)^{\frac{1}{k}} d\sigma \ll h(\ln x)^{m_k+\frac{1}{k}} \int_{\frac{1}{2}}^{\beta} \left(\frac{T^{\frac{1}{3k}}}{x}\right)^{1-\sigma} d\sigma \ll$$

$$\ll h(\ln x)^{m_k+\frac{1}{k}} \int_{\frac{1}{2}}^{1} g(\rho,\sigma) \left(\frac{T^{\frac{1}{3k}}}{x}\right)^{1-\sigma} d\sigma,$$

where

$$g(\rho,\sigma) = \begin{cases} 1, & \text{if } \sigma \leqslant \beta, \\ 0, & \text{if } \sigma > \beta. \end{cases}$$

Applying lemma 6, we get

$$\sum_{|\gamma|<T} j_\rho \ll h(\ln x)^{m_k+\frac{1}{k}} \int_{\frac{1}{2}}^{1-\varrho(T)} \left(\sum_{|\gamma|<T} g(\rho;\gamma)\right) \left(\frac{T^{\frac{1}{3k}}}{x}\right)^{1-\sigma} d\sigma \ll$$

$$\ll h(\ln x)^{m_k+\frac{1}{k}} \int_{\frac{1}{2}}^{1-\varrho(T)} N(\sigma;T) \left(\frac{T^{\frac{1}{3k}}}{x}\right)^{1-\sigma} d\sigma \ll$$

$$\ll h(\ln x)^{m_k+44+\frac{1}{k}} \int_{\frac{1}{2}}^{1-\varrho(T)} \left(\frac{T^{\frac{1}{3k}}}{x}\right)^{1-\sigma} T^{\frac{12}{5}(1-\sigma)} d\sigma \ll$$

$$\ll h(\ln x)^{45} \int_{\frac{1}{2}}^{1-\varrho(T)} \left(\frac{T^{\frac{12}{5}+\frac{1}{3k}}}{x}\right)^{1-\sigma} d\sigma \ll h(\ln x)^{45} \left(\frac{T^{\frac{12}{5}+\frac{1}{3k}}}{x}\right)^{1-\varrho(T)}.$$

Let $D(x) = e^{C_1(\ln x)^{0.8}}$. Choosing T from the equation,

$$T^{\frac{36k+5}{15k}} = xD^{-1}(x)$$

we easily conclude that

$$T = x^{\frac{15k}{36k+5}} D(x)^{-\frac{15k}{36k+5}}$$

and

$$j_0 = \frac{h}{(\ln x)^{1-\frac{1}{k}}} \left(\sum_{0 \leqslant n \leqslant N} \frac{(-1)^n \Pi_n}{\Gamma(\frac{1}{k}-n)(\ln x)^n} \right) + O(J),$$

where

$$J = \frac{x}{T}(\ln x)^{0.6} + \frac{\sqrt{x}}{\ln x} + \sqrt{x}(\ln x)^{m_k+\frac{5}{4}} + \frac{h}{(\ln x)^{N+2-\frac{1}{k}}} + \frac{h^2}{x(\ln x)^{N+2-\frac{1}{k}}} +$$

$$+ h(\ln x)^{45} \left(\frac{x^{\frac{1}{3k}}}{D^{1+\frac{1}{3k}}}\right)^{1-\varrho(T)} \ll$$

$$\ll \frac{x}{T}(\ln T)^{\frac{17}{20}} + \frac{h}{(\ln x)^{N+2-\frac{1}{k}}} + \frac{h^2}{x(\ln x)^{N+2-\frac{1}{k}}} + h(\ln x)^{45} \left(\frac{x^{\frac{1}{3k}}}{D^{1+\frac{1}{3k}}}\right)^{1-\varrho(T)}. \quad (5)$$

Obviously, the formula for S_1 is asymptotic if

$$h \gg \frac{x}{T}(\ln x)^2 = x^{\alpha_k} e^{C_2(\ln x)^{0.8}},$$

where

$$\alpha_k = 1 - \frac{15k}{36k+5} = \frac{21k+5}{36k+5}.$$

3.2 The mean-value of the function $\dfrac{\sigma(n)}{\tau(n)}$ on the short interval

Suppose that $\sigma = \operatorname{Re} s > 2$ and let

$$F(s) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{\tau_k(n)} \cdot n^{-s}.$$

This series converges absolutely, since

$$|F(s)| \leqslant \sum_{n=1}^{\infty} n^{1-\sigma} \leqslant 1 + \int_1^{\infty} u^{1-\sigma} du = 1 + \frac{1}{\sigma - 2} < +\infty$$

Setting

$$a_n = \frac{\sigma(n)}{\tau_k(n)}, \quad n \equiv A(n), b = 2 + \frac{1}{\ln x} \leqslant \frac{21}{10}, \quad \alpha = 1$$

in lemma 2, we get:

$$a_n = \frac{1}{\tau(n)} \sum_{d|n} d \leqslant \frac{n\tau(n)}{\tau(n)} \leqslant A(n),$$

$$S_2 = S(x, h; f_2) = I + O(R),$$

where

$$I = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{(x+h)^s - x^s}{s} ds, \ R = \frac{x^b}{T(b-1)} + \frac{xA(2x)\ln x}{T} \ll \frac{x^2 \ln x}{T}.$$

Further,

$$F(s) = \prod_p F_p(s),$$

where

$$F_p(s) = 1 + \sum_{k=1}^{\infty} \frac{1}{k+1} \cdot \frac{1}{p^{k(s-1)}} \cdot \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}} = 1 + \frac{1}{2p^s} + \frac{1}{2p^{s-1}} + \sum_{k=2}^{\infty} \frac{1}{k+1} \cdot \frac{1}{p^{k(s-1)}} \cdot \frac{1 - \frac{1}{p^{k+1}}}{1 - \frac{1}{p}}.$$

Obviously,

$$F_p(s+1) = 1 + \frac{p+1}{2p^{s+1}} + \frac{p^2+p+1}{3p^{2(s+1)}} + \sum_{k=3}^{\infty} \frac{p^{k+1}}{p-1} \cdot \frac{p^{-(s+1)k}}{k+1} = 1 + u(s) + v(s),$$

where

$$v(s) = \frac{1}{2p^s} + \frac{1}{3p^{2s}},$$

$$u(s) = \frac{1}{2p^{s+1}} + \frac{p+1}{3p^{2(s+1)}} + \sum_{k=3}^{\infty} \frac{p^{k+1}}{p-1} \cdot \frac{p^{-(s+1)k}}{k+1}.$$

Suppose that $p \geqslant p_0 = 5$. Then

$$|u(s)| \leqslant \frac{1}{2p^{\sigma+1}} + \frac{p+1}{3p^{2(\sigma+1)}} + \sum_{k=3}^{\infty} \frac{(k+1)p^k}{k+1} \frac{1}{p^{k(\sigma+1)}} = \frac{1}{2p^{\sigma+1}} + \frac{p+1}{3p^{2(\sigma+1)}} + \sum_{k=3}^{\infty} \frac{1}{p^{k\sigma}} =$$

$$= \frac{1}{p^{\sigma+1}}\left(\frac{1}{2}+\frac{1}{3p^{\sigma}}+\frac{1}{3p^{\sigma+1}}+\frac{1}{p^{2\sigma-1}}\cdot\frac{1}{1-\frac{1}{p^{\sigma}}}\right)\leqslant\frac{C_1}{p^{\sigma+1}},$$

where

$$C_1=\frac{1}{2}+\frac{1}{3\sqrt{p_0}}\left(1+\frac{1}{p_0}\right)+\frac{1}{1-\frac{1}{\sqrt{p_0}}},$$

$$|v(s)|\leqslant\frac{1}{2p^{\sigma}}+\frac{1}{3p^{2\sigma}}=\frac{1}{p^{\sigma}}\left(\frac{1}{2}+\frac{1}{3p^{\sigma}}\right)\leqslant\frac{C_2}{p^{\sigma}},\ C_2=\frac{1}{2}+\frac{1}{3\sqrt{p_0}}.$$

Therefore,

$$|u(s)+v(s)|\leqslant\frac{C_1}{p^{\sigma+1}}+\frac{C_2}{p^{\sigma}}=\frac{C_0}{p^{\sigma}},$$

$$C_0=\frac{C_1}{p_0}+C_2=\frac{1}{2}\left(1+\frac{1}{p_0}\right)+\frac{1}{3\sqrt{p_0}}\left(1+\frac{1}{p_0}+\frac{1}{p_0^2}\right)+\frac{1}{p_0-\sqrt{p_0}},$$

$$|u(s)\cdot v(s)|\leqslant\frac{C_1\cdot C_2}{p^{2\sigma+1}},\quad |u(s)|^2<\frac{C_1^2}{p^{2\sigma+2}},$$

where

$$C_0=\frac{C_1}{p_0}+C_2=1.1466...,\ C_1=2.4879...,\ C_2=0.6490...$$

Furthermore,

$$F_p(s+1)=\left((v+u)-\frac{1}{2}(u^2+2uv+v^2)\right)+\sum_{k\geqslant 3}\frac{(-1)^k}{k}(u+v)^k=U+V,$$

$$|V|=\left|\sum_{k\geqslant 3}\frac{(-1)^k}{k}(u+v)^k\right|\leqslant\sum_{k\geqslant 3}\frac{1}{k}\left(\frac{C_0}{p^{\sigma}}\right)^k\leqslant\frac{1}{3}\left(\frac{C_0}{p^{\sigma}}\right)^3\sum_{k=0}^{\infty}\left(\frac{C_0}{\sqrt{p_0}}\right)^k=\frac{C_3}{p^{3\sigma}},$$

where

$$C_3=\frac{C_0^3}{3}\cdot\frac{1}{1-\frac{C_0}{\sqrt{p_0}}}=1.0314...$$

Representing U in the form

$$U=(u+v)-\frac{u^2}{2}-uv-\frac{v^2}{2}=\frac{1}{2^s}+\frac{1}{3p^{2s}}+u-\frac{u^2}{2}-uv-\frac{1}{2}\left(\frac{1}{4p^{2s}}+\frac{1}{3p^{3s}}+\frac{1}{9p^{4s}}\right)=$$

$$=\frac{1}{2p^s}+\frac{5}{24}\cdot\frac{1}{p^{2s}}+W,$$

we have:

$$|W|=\left|u-\frac{u^2}{2}-uv-\frac{1}{2}\left(\frac{1}{3p^{3s}}+\frac{1}{9p^{4s}}\right)\right|\leqslant\frac{C_1}{p^{\sigma+1}}+\frac{C_1^2}{2}\cdot\frac{1}{p^{2\sigma+2}}+\frac{C_1C_2}{p^{2\sigma+1}}+\frac{1}{2}\left(\frac{1}{3p^{3\sigma}}+\frac{1}{9p^{4\sigma}}\right)\leqslant$$

$$\leqslant\frac{1}{p^{\sigma+1}}\left(C_1+\frac{C_1^2}{2}\cdot\frac{1}{p_0^{\sigma+1}}+\frac{C_1C_2}{p_0^{2\sigma+1}}+\frac{1}{2}\left(\frac{1}{3p_0^{2\sigma-1}}+\frac{1}{9p_0^{3\sigma-1}}\right)\right)\leqslant$$

$$\frac{1}{p^{\sigma+2}}\left(C_1+\frac{C_1^2}{2}\cdot\frac{1}{p_0\sqrt{p_0}}+\frac{C_1C_2}{\sqrt{p_0}}+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{9\sqrt{p_0}}\right)\right)=\frac{C_4}{p^{\sigma+1}},$$

where

$$C_4 = C_1 + \frac{1}{6} + \frac{1}{\sqrt{p_0}} \left(\frac{C_1^2}{2p_0} + C_1 C_2 \frac{1}{18} \right) = 1.7297...$$

Thus, for $p \geqslant p_0$ we obtain

$$\ln F_p(s+1) = \ln(1 + v(s) + u(s)) = \frac{1}{2p^s} + \frac{5}{24p^{2s}} + \theta_1 \left(\frac{C_3}{p^{3\sigma}} + \frac{C_4}{p^{\sigma+1}} \right) =$$

$$= \frac{1}{2p^s} + \frac{5}{24p^{2s}} + \frac{\theta_2 C_5}{p^{\sigma+1}}, \quad C_5 = C_3 + C_4 = 2.7612...$$

Now let

$$G_p(s) = F_p(s) \left(1 - \frac{1}{p^{s-1}}\right)^{\frac{1}{2}} \left(1 - \frac{1}{p^{2(s-1)}}\right)^{-\frac{1}{24}}.$$

Since

$$\frac{1}{2} \ln \left(1 - \frac{1}{p^s}\right) = -\frac{1}{2p^s} - \frac{1}{4p^{2s}} + \frac{\theta_3 C_6}{p^{3s}}, \; C_6 = \frac{1}{6} \cdot \frac{1}{1 - \frac{1}{\sqrt{p_0}}} = \frac{5 + \sqrt{5}}{24},$$

$$-\frac{1}{24} \ln \left(1 - \frac{1}{p^{2s}}\right) = \frac{1}{24p^{2s}} + \frac{5}{8 \cdot 24} \cdot \frac{\theta_4}{p^{4\sigma}},$$

then we finally get:

$$|\ln G_p(s+1)| \leqslant \frac{C_5}{p^{\sigma+1}} + \frac{C_6}{p^{3\sigma}} + \frac{5}{8 \cdot 24} \cdot \frac{1}{p^{4\sigma}} = \frac{1}{p^{\sigma+1}} \left(C_5 + \frac{C_6}{p^{2\sigma-1}} + \frac{5}{8 \cdot 24} \cdot \frac{1}{p^{3\sigma-1}} \right) \leqslant \frac{C_7}{p^{\sigma+1}},$$

$$C_7 = C_5 + C_6 + \frac{5}{8 \cdot 24} \cdot \frac{1}{\sqrt{p_0}} = 3.2387...$$

Thus,

$$|\ln G_p(s+1)| \leqslant \frac{C_7}{p^{\sigma+1}}, \quad C_7 = 3.2387...$$

It remains to check the cases of $p = 2$ and $p = 3$. For $p = 2$ we have

$$F_2(s+1) = \sum_{k=0}^{\infty} \frac{2^{k+1} - 1}{k+1} 2^{-k(s+1)} = 2^{s+1} \sum_{k=0}^{\infty} \frac{2^{k+1} - 1}{k+1} 2^{-(k+1)(s+1)} =$$

$$= 2^{s+1} \left(\sum_{k=0}^{\infty} \frac{2^{k+1} \cdot 2^{-(k+1)(s+1)}}{k+1} - \sum_{k=0}^{\infty} \frac{2^{-(k+1)(s+1)}}{k+1} \right) =$$

$$= 2^{s+1} \left(-\ln \left(1 - \frac{1}{2^s}\right) - \ln \left(1 - \frac{1}{2^{s+1}}\right) \right) = 2^{s+1} \varphi_2(s),$$

where

$$\varphi_p(s) = \ln \frac{p^{s+1} - 1}{p^{s+1} - p}.$$

Similarly, in case of $p = 3$ we get

$$F_3(s+1) = \frac{3^{s+1}}{2} \, \varphi_3(s).$$

Since $p^{s+1}-1 \neq p^{s+1}-p$, then $\varphi_p(s) \neq 0$. As $\varphi_p\left(s+\frac{2\pi i}{\ln p}\right) = \varphi_p(s)$, then the maximum and minimum of $|\varphi_p(s)|$ in the strip $\frac{1}{2} \leqslant \operatorname{Re} s \leqslant \frac{11}{10}$ coincide with the maximum and minimum of $|\varphi_p(s)|$ in rectangular

$$\frac{1}{2} \leqslant \operatorname{Re} s \leqslant \frac{11}{10}, \quad 0 \leqslant \operatorname{Im} s \leqslant \frac{2\pi}{\ln p}. \tag{6}$$

Since $\varphi(s)$ is analytic in (6) and $\varphi_p(s) \neq 0$ in (6), then, according to the maximum principle (see [14], ch. 32) the extremal values $|\varphi_p(s)|$ have to be reached on the boundary of (6). Since $\varphi_p(s)$ is periodic, then it is enough to examine the behaviour of s along the three sides of (6).

By calculations,

$$\max_{\substack{0 \leqslant t \leqslant 2\pi/\ln 2\\ \sigma=1/2}} |\varphi_2(s)| = \left|\varphi_2\left(\tfrac{1}{2}\right)\right| = 0.79168...; \quad \min_{\substack{0 \leqslant t \leqslant 2\pi/\ln 2\\ \sigma=1/2}} |\varphi_2(s)| = 0.23206...$$

$$\max_{\substack{0 \leqslant t \leqslant 2\pi/\ln 2\\ \sigma=11/10}} |\varphi_2(s)| = \left|\varphi_2\left(\tfrac{11}{10}\right)\right| = 0.36279...; \quad \min_{\substack{0 \leqslant t \leqslant 2\pi/\ln 2\\ \sigma=11/10}} |\varphi_2(s)| = 0.17323...$$

$$\max_{\substack{1/2 \leqslant \sigma \leqslant 11/10\\ t=0}} |\varphi_2(s)| = \left|\varphi_2\left(\tfrac{1}{2}\right)\right| = 0.79168...; \quad \min_{\substack{1/2 \leqslant \sigma \leqslant 11/10\\ t=0}} |\varphi_2(s)| = 0.36272...$$

Then, in (6) we have

$$A_1 \leqslant |\varphi_2(s)| \leqslant B_1, \quad A_1 = 0.17323..., \quad B_1 = 0.79168...$$

Similarly, for $\varphi_3(s)$ we get

$$\max_{\substack{0 \leqslant t \leqslant 2\pi/\ln 3\\ \sigma=1/2}} |\varphi_3(s)| = \left|\varphi_3\left(\tfrac{1}{2}\right)\right| = 0.64746...; \quad \min_{\substack{0 \leqslant t \leqslant 2\pi/\ln 3\\ \sigma=1/2}} |\varphi_3(s)| = 0.279736...$$

$$\max_{\substack{0 \leqslant t \leqslant 2\pi/\ln 3\\ \sigma=11/10}} |\varphi_3(s)| = \left|\varphi_3\left(\tfrac{11}{10}\right)\right| = 0.24985...; \quad \min_{\substack{0 \leqslant t \leqslant 2\pi/\ln 3\\ \sigma=11/10}} |\varphi_3(s)| = 0.16642...$$

$$\max_{\substack{1/2 \leqslant \sigma \leqslant 11/10\\ t=0}} |\varphi_3(s)| = \left|\varphi_3\left(\tfrac{1}{2}\right)\right| = 0.64746...; \quad \min_{\substack{1/2 \leqslant \sigma \leqslant 11/10\\ t=0}} |\varphi_3(s)| = 0.24989...$$

Consequently, in (6) we have

$$A_2 \leqslant |\varphi_3(s)| \leqslant B_2, \quad A_2 = 0.16642..., \quad B_2 = 0.64746...$$

Now let us estimate the functions

$$G_p(s+1) = F_p(s+1)\left(1-\frac{1}{p^s}\right)^{\frac{1}{2}}\left(1-\frac{1}{p^{2s}}\right)^{-\frac{1}{24}}, \quad \text{for} \quad p = 2, 3.$$

If $p = 2$ then

$$|G_p(s+1)| \leqslant 2^{\sigma+1}\left(1+\frac{1}{2^\sigma}\right)^{\frac{1}{2}}\left(1-\frac{1}{2^{2\sigma}}\right)^{-\frac{1}{24}} B_1 =$$

$$= \frac{2\sqrt{2^{2\sigma}+2^\sigma}}{\left(1-\frac{1}{2^{2\sigma}}\right)^{\frac{1}{24}}} B_1 \leqslant \frac{2\sqrt{2^{2\cdot 2}+2^{1.1}}}{\left(1-\frac{1}{2^{2\cdot 2}}\right)^{\frac{1}{24}}} B_1 < 2.245028 \cdot B_1 < 4.1524;$$

$$|G_p(s+1)| > 2^{\sigma+1}\left(1-\frac{1}{2^\sigma}\right)^{\frac12}\left(1+\frac{1}{3^{2\sigma}}\right)^{-\frac{1}{24}} A_1 =$$

$$= \frac{2\sqrt{2^{2\sigma}-2^\sigma}}{\left(1+\frac{1}{2^{2\sigma}}\right)^{\frac{1}{24}}}A_1 > \frac{2\sqrt{2-\sqrt{2}}}{\left(\frac32\right)^{\frac{1}{24}}}A_1 > 1.505090\cdot A_1 > 0.260726.$$

In case $p=3$ we have

$$|G_p(s+1)| \leqslant \frac{3^{\sigma+1}}{2}\left(1+\frac{1}{3^\sigma}\right)^{\frac12}\left(1-\frac{1}{3^{2\sigma}}\right)^{-\frac{1}{24}} B_2 =$$

$$= \frac{\frac32\sqrt{3^{2\sigma}+3^\sigma}}{\left(1-\frac{1}{3^{2\sigma}}\right)^{\frac{1}{24}}}B_2 \leqslant \frac{\frac32\sqrt{3^{2\cdot 2}+3^{1.1}}}{\left(1-\frac{1}{3^{2\cdot 2}}\right)^{\frac{1}{24}}}B_2 < 5.745949 B_2 < 3.720278;$$

$$|G_p(s+1)| > \frac{3^{\sigma+1}}{2}\left(1-\frac{1}{3^\sigma}\right)^{\frac12}\left(1+\frac{1}{3^{2\sigma}}\right)^{-\frac{1}{24}} A_2 =$$

$$= \frac{\frac32\sqrt{3^{2\sigma}-3^\sigma}}{\left(1+\frac{1}{3^{2\sigma}}\right)^{\frac{1}{24}}}A_2 > \frac{\frac32\sqrt{3-\sqrt{3}}}{\left(\frac43\right)^{\frac{1}{24}}}\cdot A_2 > 1.668923\cdot A_2 > 0.277752.$$

Thus,

$$|G(s+1)| = \prod_p |G_p(s+1)| \leqslant |G_2(s+1)|\cdot|G_3(s+1)|\cdot \exp\left(\sum_p \ln|G_p(s+1)|\right) \leqslant$$

$$\leqslant 4.1524\cdot 3.720278\cdot \exp\left(\sum_{p\geqslant 5}\frac{C_7}{p^{3/2}}\right) < 15.4481\cdot e^{0.1605\cdot C_7} < 15.4481\cdot e^{0.52} < 26,$$

and

$$|G(s+1)| > 0.260726\cdot 0.277752\cdot e^{-0.52} > 0.04305 > \frac{1}{24}.$$

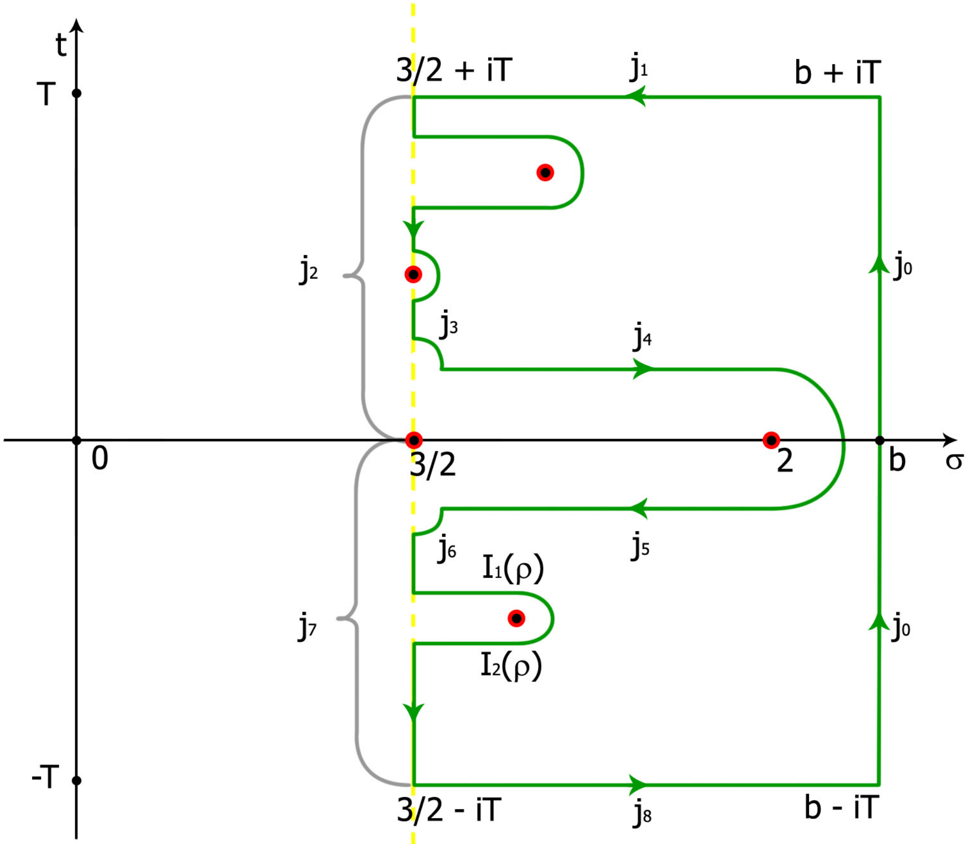
Finally we have $\frac{1}{24} < |G(s)| < 26$ in the strip $\frac32 \leqslant \sigma \leqslant \frac{21}{10}$,.

Now let us consider the integral I. Let Γ be the boundary of the rectangle with the vertices $\frac32 \pm iT, b\pm iT$, $2\leqslant T\leqslant x$, where the zeros of $\zeta(s-1)$ of the form $\rho = \frac32 + i\gamma$, $|\gamma| < T$, are avoided by the semicircles of the infinitely small radius lying to the right of the line $\operatorname{Re} s = \frac32$, the pole of $\zeta(s-1)$ at the point $s=2$ is avoided be the arcs Γ_1, Γ_2 with the radius $\frac{1}{\ln x}$, and let a horizontal cut be drawn from the critical line inside this rectangle to each zero $\rho = \beta + i\gamma, \frac32 < \beta < 2, |\gamma| < T$. Then the function $|F(s)|$ is analytic inside Γ. By the Cauchy residue theorem,

$$j_0 = -\sum_{k=1}^{8} j_k - \sum_\rho j_\rho = -(j_4+j_5) - \sum_{k\neq 4,5} j_k - \sum_\rho j_\rho.$$

By the proof of theorem 1, we have on Γ:

$$F(s) = \frac{(\zeta(s-1))^{\frac12}}{\zeta(2(s-1))^{\frac{1}{24}}}\cdot G(s).$$



According to the lemma 5,

$$|F(s)| \ll T^{\frac{1-(\sigma-1)}{6}}(\ln T)^{\frac{13}{24}} = T^{\frac{2-\sigma}{6}}(\ln T)^{\frac{13}{24}}.$$

Then

$$|j_1| \ll \frac{1}{T}\int_{\frac{3}{2}}^{-b} x^{\sigma} T^{\frac{2-\sigma}{6}}(\ln T)^{\frac{13}{24}} d\sigma \ll \frac{x^2}{T}\int_{\frac{3}{2}}^{-b}\left(\frac{x}{T^{\frac{1}{6}}}\right)^{\sigma-2}(\ln T)^{\frac{13}{24}} d\sigma \ll \frac{x^2}{T}(\ln T)^{\frac{13}{24}}.$$

The similar estimate is valid for j_8.

Next, on the arcs Γ_1 and Γ_2 we have

$$|\zeta(s-1)| = \left|\frac{1}{2} + \frac{1}{s-2} + (s-1)\int_1^{\infty}\frac{\rho(u)}{u^s}du\right| \leqslant 0.5 + 1 + 0.1 + 1.6\cdot 0.5\int_1^{\infty}\frac{du}{u^{\frac{3}{2}}} < 3.2,$$

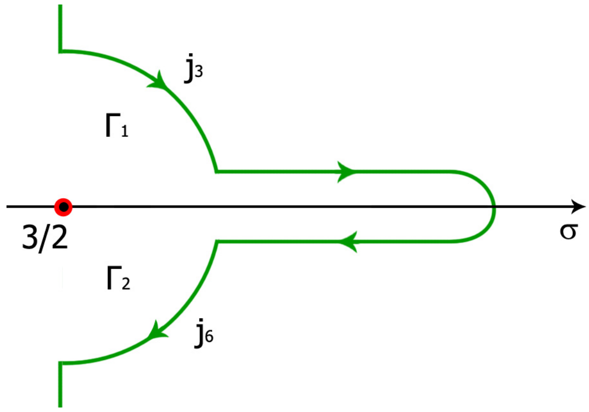
$$|\zeta(2(s-1))| \geqslant \frac{1}{|2s-3|} - 1.1 \geqslant 0.5\ln x - 1.1 > 0.4\ln x.$$

Hence

$$|F(s)| \leqslant \frac{(3.2)^{\frac{1}{2}}}{(0.4\ln x)^{\frac{1}{24}}} < 1.$$

Therefore,

$$|j_3 + j_6| \leqslant \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{2(2x)^{\frac{3}{2}+\frac{1}{\ln x}}}{\frac{5}{2}}\frac{d\varphi}{\ln x} \ll \frac{x^{\frac{3}{2}}}{\ln x}.$$



Since

$$|F(s)| \leqslant |\zeta(s-1)|^{\frac{1}{2}} (\ln x)^{\frac{1}{24}} |G(s)| \ll (\ln x)^{\frac{1}{24}} \left|(\zeta(\sigma - 1 + it))\right|^{\frac{1}{2}},$$

then

$$|j_2| = \left| \text{p.v.} \frac{1}{2\pi i} \int_{\frac{3}{2}+iT}^{\frac{3}{2}+\frac{i}{\ln x}} F(s) \cdot \frac{(x+h)^s - x^s}{s} ds \right| \ll \int_{\frac{1}{\ln x}}^{T} (\ln x)^{\frac{1}{24}} \cdot |\zeta(\sigma - 1 + it)|^{\frac{1}{2}} x^{\frac{3}{2}} \frac{dt}{t+1} \ll$$

$$\ll (\ln x)^{\frac{1}{24}} x^{\frac{3}{2}} \int_0^T \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2}} dt}{t+1} = (\ln x)^{\frac{1}{24}} x^{\frac{3}{2}} \sum_{\nu \geqslant 0} \int_{T/2^\nu}^{T/2^{\nu+1}} \frac{\left|\zeta\left(\frac{1}{2}+it\right)\right|^{\frac{1}{2}}}{t+1} dt.$$

Denoting the summands in the last sum by $j(\nu)$ and taking $X = T \cdot 2^{-\nu}$, by Hölder inequality we get:

$$j(\nu) \ll \frac{1}{X} \left(\int_X^{2X} \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^2 dt \right)^{\frac{1}{4}} X^{1-\frac{1}{4}} \ll \frac{1}{X} (X \ln X)^{\frac{1}{4}} X^{\frac{3}{4}} \ll (\ln X)^{\frac{1}{4}} \ll (\ln T)^{\frac{1}{4}}.$$

Thus,

$$\sum_{\nu \geqslant 0} j(\nu) \ll (\ln T)^{\frac{5}{4}}.$$

Then

$$|j_2| \ll (\ln x)^{\frac{1}{24}} x^{\frac{3}{2}} (\ln x)^{\frac{5}{4}} \ll (\ln x)^{\frac{31}{24}} x^{\frac{3}{2}}.$$

The integral j_7 is estimated as above.

The main term arises from j_4 and j_5. Let us define the entire function $w(s)$ by the relation

$$\zeta(s-1) = \frac{w(s-1)}{s-2}.$$

Let $s = 2 - u + i \cdot 0$, where $0 \leqslant u \leqslant \frac{1}{2}$. Then

$$\sqrt{\zeta(s-1)} = \frac{\sqrt{w(s-1)}}{\sqrt{-u + i \cdot 0}}.$$

Since $-u + i\varepsilon \to u \cdot e^{\pi i}$ as $\varepsilon \to +0$ then

$$\sqrt{-u + i \cdot 0} = \sqrt{u} e^{\frac{\pi i}{2}}, \quad \sqrt{\zeta(s-1)} = \frac{\sqrt{w(\sigma - 1)}}{\sqrt{u}} e^{-\frac{\pi i}{2}}.$$

Therefore, on the upper edge of the cut we have

$$F(s)=\frac{\sqrt{w(1-u)}}{(\zeta(2-2u))^{\frac{1}{24}}}\,G(2-u)\,\frac{e^{-\frac{\pi i}{2}}}{\sqrt{u}}=\frac{\Pi(u)e^{-\frac{\pi i}{2}}}{\sqrt{u}}=\frac{\Pi(u)}{i\sqrt{u}},$$

where

$$\Pi(u)=G(2-u)\frac{\sqrt{w(1-u)}}{(\zeta(2-2u))^{\frac{1}{24}}}.$$

Then for j_4 we obtain:

$$\begin{aligned}
j_4&=\frac{1}{2\pi i}\int_{\frac{3}{2}+\frac{1}{\ln x}+i\cdot 0}^{2+i\cdot 0}F(\sigma+i\cdot 0)\frac{(x+h)^s-x^s}{s}ds=\\
&=\frac{1}{2\pi i}\int_{\frac{1}{2}}^{0}\frac{\Pi(u)}{i\sqrt{u}}\frac{(x+h)^{2-u}-x^{2-u}}{2-u}(-du)=\\
&=-\frac{1}{2\pi}\int_0^{\frac{1}{2}}\frac{\Pi(u)}{\sqrt{u}}\int_x^{x+h}y^{1-u}dydu=\\
&=\frac{1}{2\pi}\int_x^{x+h}y\int_0^{\frac{1}{2}}\frac{\Pi(u)}{\sqrt{u}}y^{-u}dudy.
\end{aligned}$$

Suppose that $N\geqslant 0$ is fixed. Then

$$\Pi(u)=\Pi_0+\Pi_1u+\Pi_2u^2+\ldots+\Pi_Nu^N+O_N(u^{N+1})$$

and

$$\begin{aligned}
j_4&=-\frac{1}{2\pi}\int_x^{x+h}y\left(\sum_{0\leqslant n\leqslant N}\Pi_n\int_0^{\frac{1}{2}}\frac{u^ny^{-u}}{\sqrt{u}}du\right)dy+O\left(J\right)=\\
&=-\frac{1}{2\pi}\sum_{0\leqslant n\leqslant N}\Pi_n\Gamma\left(n+\frac{1}{2}\right)\int_x^{x+h}\frac{y\;dy}{(\ln y)^{n+\frac{1}{2}}}+O\left(J\right),
\end{aligned}$$

where

$$J=-\frac{1}{2\pi}\int_x^{x+h}y\int_0^{\frac{1}{2}}\frac{u^{N+1}y^{-u}}{\sqrt{u}}du\;dy\ll\frac{xh}{(\ln x)^{N+\frac{3}{2}}}\Gamma\left(N+\frac{3}{2}\right).$$

Let us evaluate a contribution of n-th term to the sum. Assume

$$\varphi(y)=\frac{y}{(\ln y)^{\nu}},\quad \varphi'(y)=\frac{y\ln y-\nu}{y(\ln y)^{\nu+1}}.$$

If $x\leqslant y\leqslant x+h$ then

$$\varphi(y)=\varphi(x)+(y-x)\;\theta\;\frac{x\ln x+\nu}{x(\ln x)^{\nu+1}},\quad |\theta|\leqslant 1.$$

Hence,

$$\int_x^{x+h}\varphi(y)dy=h\varphi(x)+\theta_1\frac{x\ln x+\nu}{x(\ln x)^{\nu+1}}\int_x^{x+h}(y-x)dy=\frac{xh}{(\ln x)^{\nu}}+\frac{\theta_1}{2}\,h^2\,\frac{x\ln x+\nu}{x(\ln x)^{\nu+1}}.$$

Finally, we state:

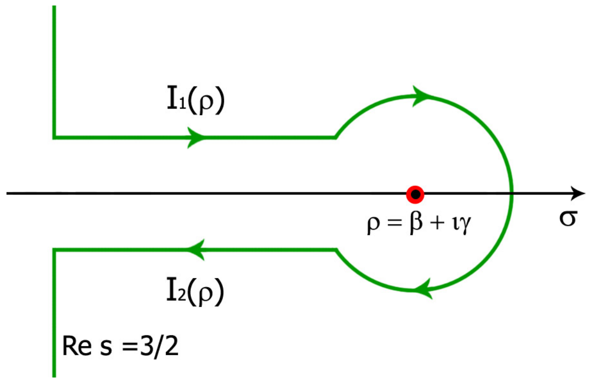
$$j_4 = -\frac{xh}{2\pi}\sum_{0\leqslant n\leqslant N}\frac{\Pi_n\Gamma\left(n+\frac{1}{2}\right)}{(\ln x)^{n+\frac{1}{2}}} + O\left(\frac{h}{(\ln x)^{N+\frac{3}{2}}}\right) + O\left(\frac{h^2}{x}\frac{1}{(\ln x)^{N+\frac{3}{2}}}\right)$$

and

$$-(j_4+j_5) = -\frac{xh}{2\pi(\ln x)^{\frac{1}{2}}}\left(\sum_{0\leqslant n\leqslant N}\frac{\Pi_n\Gamma\left(n+\frac{1}{2}\right)}{(\ln x)^{n}} + O\left(\frac{1}{(\ln x)^{N+1}}\right) + O\left(\frac{h}{x(\ln x)^{N+1}}\right)\right).$$

Now let us estimate the sum

$$\sum_{|\gamma|<T} j_\rho, \quad \text{where} \quad j_\rho = I_1(\rho) + I_2(\rho).$$



By analogy with the proof of the theorem 1, we get

$$I_1(\rho) = \int_{\frac{3}{2}+i\gamma}^{\beta+1+i\gamma} F(s)\frac{(x+h)^s - x^s}{s}ds \ll \int_{\frac{3}{2}}^{\beta+1} |F(s)|\, hx^{\sigma-1}d\sigma \ll$$

$$\ll h\int_{\frac{3}{2}}^{\beta+1} T^{\frac{2-\sigma}{6}}(\ln x)^{\frac{13}{24}}x^{\sigma-1}d\sigma \ll xh(\ln x)^{\frac{13}{24}}\int_{\frac{3}{2}}^{\beta+1}\left(\frac{T^{\frac{1}{6}}}{x}\right)^{2-\sigma}d\sigma.$$

Hence,

$$|j_\rho| \ll xh(\ln x)^{\frac{13}{24}}\int_{\frac{3}{2}}^{\beta+1}\left(\frac{T^{\frac{1}{6}}}{x}\right)^{2-\sigma}d\sigma \ll xh(\ln x)^{\frac{13}{24}}\int_{\frac{1}{2}}^{\beta}\left(\frac{T^{\frac{1}{6}}}{x}\right)^{1-v}dv =$$

$$= xh(\ln x)^{\frac{13}{24}}\int_{\frac{1}{2}}^{1} g(\rho;v)\left(\frac{T^{\frac{1}{6}}}{x}\right)^{1-v}dv,$$

where

$$g(\rho;v) = \begin{cases} 1, & \text{if } v\leqslant\beta, \\ 0, & \text{if } v>\beta. \end{cases}$$

Applying the lemma 6, we obtain:

$$\sum_{|\gamma|<T} j_\rho \ll xh(\ln x)^{\frac{13}{24}} \int_{\frac{1}{2}}^{1-\varrho(T)} \left(\sum_{|\gamma|<T} g(\rho;v) \right) \left(\frac{T^{\frac{1}{6}}}{x} \right)^{1-v} dv \ll$$

$$\ll xh(\ln x)^{\frac{13}{24}} \int_{\frac{1}{2}}^{1-\varrho(T)} N(v;T) \left(\frac{T^{\frac{1}{6}}}{x} \right)^{1-v} dv \ll xh(\ln x)^{44+\frac{13}{24}} \left(\frac{T^{\frac{12}{5}+\frac{1}{6}}}{x} \right)^{1-\varrho(T)} \ll$$

$$\ll xh(\ln x)^{45} \left(\frac{T^{\frac{77}{30}}}{x} \right)^{1-\varrho(T)}.$$

Choosing T from the equation

$$T^{\frac{77}{30}} = x \cdot D^{-1}(x), \ D(x) = e^{C_2(\ln x)^{0.8}},$$

we get

$$T = x^{\frac{30}{77}} D(x)^{-\frac{30}{77}}.$$

Now we can easily conclude that the formula

$$j_0 = \frac{xh}{2\pi(\ln x)^{\frac{1}{2}}} \left(\sum_{0 \leqslant n \leqslant N} \frac{\Pi_n \Gamma(n+\frac{1}{2})}{(\ln x)^n} \right) + O(J),$$

where

$$J = \frac{x^2}{T}(\ln T)^{\frac{13}{24}} + x^{\frac{3}{2}}(\ln x)^{\frac{31}{24}} + \frac{x^{\frac{3}{2}}}{\ln x} + \frac{xh}{(\ln x)^{N+\frac{3}{2}}} + \frac{h^2}{(\ln x)^{N+\frac{3}{2}}} + xh(\ln x)^{45} \left(\frac{T^{\frac{77}{30}}}{x} \right)^{1-\varrho(T)} \ll$$

$$\ll \frac{x^2}{T}(\ln T)^{\frac{5}{6}} + \frac{xh}{(\ln x)^{N+\frac{3}{2}}} + \frac{h^2}{(\ln x)^{N+\frac{3}{2}}} + xh(\ln x)^{45} \left(\frac{T^{\frac{77}{30}}}{x} \right)^{1-\varrho(T)},$$

is an asymptotic, if

$$h = x^\alpha e^{C_2(\ln x)^{0.8}} \gg \frac{x}{T}(\ln x)^2,$$

where

$$\alpha = 1 - \frac{30}{77} = \frac{47}{77}.$$

3.3 The mean-value of the function $\frac{1}{r(n)}$ on the short interval

Suppose that $\sigma = \operatorname{Re} s > 1$ and let

$$\varkappa(n) = \sum_{d|n} \chi_4(d) = \frac{1}{4}\, r(n), \quad F(s) = \sum_{n=1}^{\infty}{}' \frac{1}{\varkappa(n)} \cdot n^{-s},$$

where $\sum'$ means that the summing is going over all n, with $\varkappa(n) \neq 0$. This series converges absolutely, since

$$|F(s)| \leqslant \sum_{n=1}^{\infty}{}' |a_n| \cdot n^{-\sigma} \leqslant \sum_{n=1}^{\infty}{}' n^{-\sigma} \leqslant 1 + \int_1^{\infty} \frac{du}{u^{\sigma}} = 1 + \frac{1}{\sigma - 1}.$$

Setting $a_n = \dfrac{1}{\varkappa(n)}$, $A(n) \equiv 1$, $b = 1 + \dfrac{1}{\ln x}$, $\alpha = 1$ in the lemma 2, we get:

$$S_3 = S(x, h; f_3) = I + O(R),$$

where

$$I = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} F(s) \frac{x^s}{s} ds, \; R = \frac{x^b}{T(b-1)} + \frac{xA(2x)\ln x}{T} \ll \frac{x \ln x}{T}.$$

Further, let $F(s) = \prod_p{}' F_p(s)$. Then

$$F_2(s) = 1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \ldots = \frac{1}{1 - \frac{1}{2^s}} = \left(1 - \frac{1}{2^s}\right)^{-1}.$$

In the case $p \equiv 1 (\operatorname{mod} 4)$, we have

$$F_p(s) = 1 + \frac{1}{2p^s} + \frac{1}{3p^{2s}} + \ldots = \left(1 - \frac{1}{p^s}\right)^{-\frac{1}{2}} \left(1 - \frac{1}{p^{2s}}\right)^{\frac{1}{24}} G_p(s),$$

where the function $G_p(s)$ is defined in the proof ot the theorem 1. Finally, for $p \equiv 3 (\operatorname{mod} 4)$ it is true that

$$F_p(s) = 1 + \frac{1}{p^{2s}} + \frac{1}{p^{4s}} + \ldots = \left(1 - \frac{1}{p^{2s}}\right)^{-1}.$$

Let us write $F(s)$ in the form

$$F(s) = \left(1 - \frac{1}{2^s}\right)^{-1} \prod_{p \equiv 1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^s}\right)^{-\frac{1}{2}} \left(1 - \frac{1}{p^{2s}}\right)^{\frac{1}{24}} G(s) \prod_{p \equiv 3 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^{2s}}\right)^{-1},$$

where

$$G(s) = \prod_p G_p(s) = \prod_{p \equiv 1 (\operatorname{mod} 4)} \left(1 - \frac{1}{p^s}\right)^{\frac{1}{2}} \left(1 - \frac{1}{p^{2s}}\right)^{-\frac{1}{24}} \left(1 + \frac{1}{2p^s} + u(s)\right),$$

$$u(s) = \frac{1}{3p^{2s}} + \ldots$$

Applying the arguments used in the proof of Theorem 1 in the case $k = 2$, we get

$$|\ln G_p(s)| \leqslant \frac{19}{2} \cdot \frac{1}{p^{3\sigma}} \leqslant \frac{19}{2p^{\frac{3}{2}}}.$$

Thus,

$$|\ln G(s)| \leqslant \frac{19}{2} \sum_{p \equiv 1 (\operatorname{mod} 4)} \frac{1}{p^{\frac{3}{2}}} \leqslant \frac{19}{2} \sum_{p \geqslant 5} \frac{1}{p^{\frac{3}{2}}} < 1.525,$$

$$e^{-1.525} < |G(s)| < e^{1.525}, \;\; 0.2 < |G(s)| < 4.6.$$

Let

$$f(s) = \prod_{p\equiv 1(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Since

$$\frac{1+\chi_4(p)}{2} = \begin{cases} 1, \text{ if } p \equiv 1(\mathrm{mod}\, 4), \\ 0, \text{ if } p \equiv 3(\mathrm{mod}\, 4), \\ \frac{1}{2}, \text{ if } p \text{ is an odd number}, \end{cases}$$

then

$$\ln f(s) = -\sum_{p\equiv 1(\mathrm{mod}\, 4)} \ln\left(1 - \frac{1}{p^s}\right) = -\sum_p \frac{1+\chi_4(p)}{2} \ln\left(1 - \frac{1}{p^s}\right) + \frac{1}{2}\ln\left(1 - \frac{1}{2^s}\right) =$$

$$= \frac{1}{2}\ln\zeta(s) + \frac{v(s)}{2} + \frac{1}{2}\ln\left(1 - \frac{1}{2^s}\right),$$

where

$$v(s) = -\sum_p \chi_4(p)\ln\left(1 - \frac{1}{p^s}\right).$$

Using the relation

$$\ln L(s,\chi_4) = -\sum_p \ln\left(1 - \frac{\chi_4(p)}{p^s}\right),$$

we obtain

$$v(s) = \sum_p \left(\ln\left(1 - \frac{\chi_4(p)}{p^s}\right) - \chi_4(p)\ln\left(1 - \frac{1}{p^s}\right) - \ln\left(1 - \frac{\chi_4(p)}{p^s}\right)\right) = \ln L(s,\chi_4) + w(s),$$

where

$$w(s) = \sum_p w_p(s), \;\; w_p(s) = \ln\left(1 - \frac{\chi_4(p)}{p^s}\right) - \chi_4(p)\ln\left(1 - \frac{1}{p^s}\right),$$

with

$$w_p(s) = \begin{cases} 0, \text{ if } p = 2, p \equiv 1(\mathrm{mod}\, 4), \\ \ln\left(1 - \frac{1}{p^{2s}}\right), \text{ if } p \equiv 3(\mathrm{mod}\, 4). \end{cases}$$

Then for $v(s)$ it holds that:

$$v(s) = \ln L(s,\chi_4) + \sum_{p\equiv 3(\mathrm{mod}\, 4)} \ln\left(1 - \frac{1}{p^{2s}}\right).$$

Hence,

$$\ln f(s) = \frac{1}{2}\ln\zeta(s) + \frac{1}{2}\ln L(s,\chi_4) + \frac{1}{2}\sum_{p\equiv 3(\mathrm{mod}\, 4)} \ln\left(1 - \frac{1}{p^{2s}}\right) + \frac{1}{2}\ln\left(1 - \frac{1}{2^s}\right),$$

Thus,

$$F(s)=\left(1-\frac{1}{2^s}\right)^{-1}(f(s))^{\frac{1}{2}}(f(2s))^{-\frac{1}{24}}G(s)\prod_{p\equiv 3(\mathrm{mod}\,4)}\left(1-\frac{1}{p^{2s}}\right)^{-1}=$$

$$=\left(1-\frac{1}{2^s}\right)^{-1}(\zeta(s)L(s,\chi_4))^{\frac{1}{4}}\prod_{p\equiv 3(\mathrm{mod}\,4)}\left(1-\frac{1}{p^{2s}}\right)^{-\frac{3}{4}}\left(1-\frac{1}{2^s}\right)^{\frac{1}{4}}G(s)\,(f(2s))^{-\frac{1}{24}}=$$

$$=\left(1-\frac{1}{2^s}\right)^{-\frac{3}{4}}(\zeta(s)L(s,\chi_4))^{\frac{1}{4}}(\zeta(2s))^{\frac{3}{4}}\left(1-\frac{1}{2^{2s}}\right)^{\frac{3}{4}}(f(2s))^{-\frac{19}{24}}G(s)=$$

$$=(\zeta(s)L(s,\chi_4))^{\frac{1}{4}}(\zeta(2s))^{\frac{17}{48}}(L(2s,\chi_4))^{-\frac{19}{48}}H(s),$$

where

$$H(s)=\left(1-\frac{1}{2^s}\right)^{-\frac{3}{4}}\left(1-\frac{1}{2^{2s}}\right)^{\frac{17}{48}}\prod_{p\equiv 3(\mathrm{mod}\,4)}\left(1-\frac{1}{p^{4s}}\right)^{-\frac{19}{48}}G(s).$$

For $H(s)$ we have:

$$\ln H(s)=-\frac{3}{4}\ln\left(1-\frac{1}{2^s}\right)+\frac{17}{48}\ln\left(1-\frac{1}{2^{2s}}\right)+\ln G(s)-\frac{19}{48}\sum_{p\equiv 3(\mathrm{mod}\,4)}\ln H_p(s),$$

where

$$H_p(s)=1-\frac{1}{p^{4s}}.$$

Then

$$|\ln H_p(s)|\leqslant\sum_{k=0}^{\infty}\left|\frac{1}{p^{4s(k+1)}(k+1)}\right|\leqslant\frac{1}{p^{4\sigma}}\frac{1}{1-\frac{1}{p^{4\sigma}}}\leqslant\frac{1}{p^{4\sigma}}\frac{1}{1-\frac{1}{p^2}}\leqslant\frac{2}{p^{4\sigma}}.$$

Next

$$|\ln H(s)|\leqslant C_1+\frac{19}{24}\sum_{p\equiv 3(\mathrm{mod}\,4)}\frac{1}{p^{4\sigma}}\leqslant C_2,$$

so that

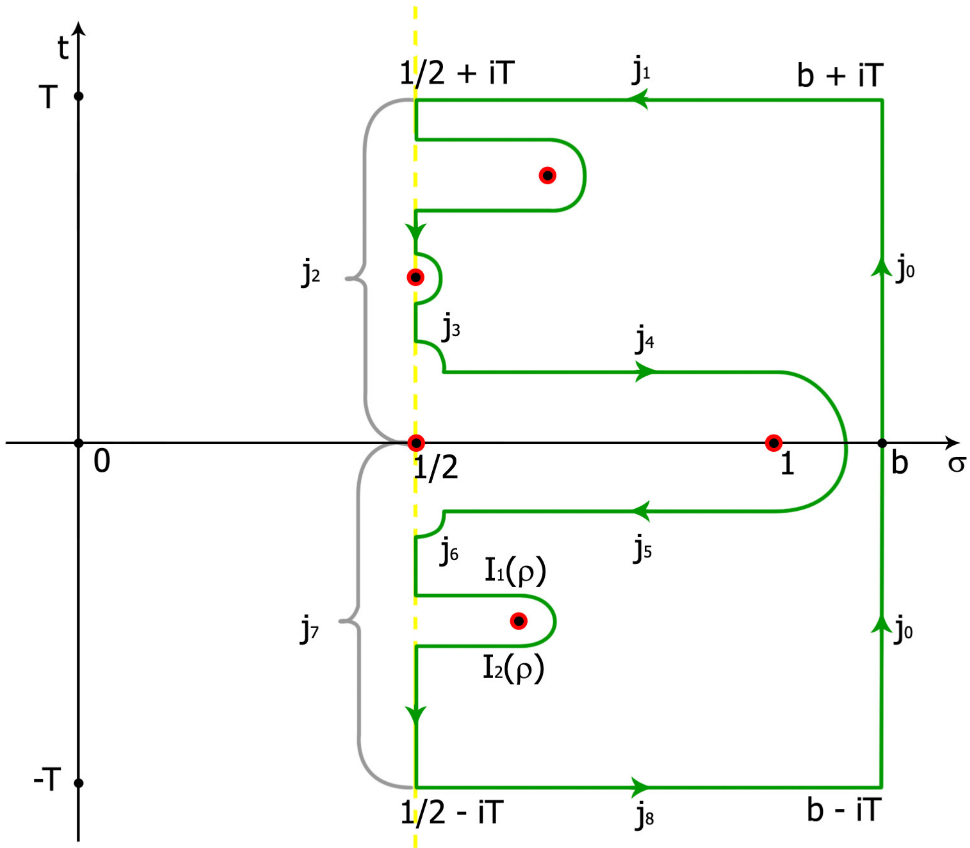
$$e^{-C_2}\leqslant|H(s)|\leqslant e^{C_2}.$$

Suppose that $2\leqslant T\leqslant x$ does not coincide with the ordinate of zeros of $\zeta(s)$ and $L(s,\chi_4)$. Let Γ be the boundary of the rectangle with the vertices $\frac{1}{2}\pm iT, b\pm iT$, where the zeros of $\zeta(s)$ and $L(s,\chi_4)$ are avoided by semicircles of the infinitely small radius lying to the right of the line $\mathrm{Re}\,s=\frac{1}{2}$, the point $s=\frac{1}{2}$ is avoided by two arcs Γ_1,Γ_2 with the radius $\frac{1}{\ln x}$, and let a horizontal cut be drawn from the critical line inside this rectangle to each zero $\rho=\frac{1}{2}+i\gamma, \frac{1}{2}<\beta<1, |\gamma|<T$. Then the function $F(s)$ is analytic inside Γ. Applying the Cauchy residue theorem,

$$j_0=-\sum_{k=1}^{8}j_k-\sum_{\rho}j_\rho=-(j_4+j_5)-\sum_{k\neq 4,5}j_k-\sum_{\rho}j_\rho.$$

Consider the integrals j_1 and j_8. For $\mathrm{Re}\,s\geqslant\frac{1}{2}+\frac{1}{\ln x}$ we have:

$$|\zeta(2s)|\leqslant 1+\sum_{n=2}^{\infty}\frac{1}{n^{2\sigma}}\leqslant 1+\frac{1}{2\sigma-1}\leqslant 1+\frac{\ln x}{2}<\ln x.$$



Suppose that $|s-1| \leqslant \frac{1}{\ln x}$. If $\operatorname{Re} s > 1$, then

$$L(s,\chi_4) = \sum_{n=1}^{\infty} \frac{\chi_4(n)}{n^s}.$$

This series converges absolutely. Then,

$$L'(s,\chi_4) = -\sum_{n=2}^{\infty} \frac{\chi_4(n)\ln n}{n^s}.$$

By Abel summation formula,

$$L^{'}(s,\chi_4) = -s\int_1^{\infty} \frac{\mathbb{C}(u)}{u^{s+1}}du, \tag{7}$$

where

$$\mathbb{C}(u) = \sum_{1<n\leqslant u} \chi_4(u)\ln u.$$

The integral in (7) converges for $\operatorname{Re} s > 0$. It means that we can use (7) for $|s-1| \leqslant \frac{1}{\ln x}$. Let us note that

$$|\mathbb{C}(u)| < \ln u,$$

uniformly in $u \geqslant 1$. Indeed, if $u = 4m+1$, then

$$\mathbb{C}(u) = \ln(4m+1) - (\ln(4m-1) - \ln(4m-3)) - (\ln(4m-5) - \ln(4m-7)) - \ldots < \ln(4m+1) < \ln u,$$

$$\mathbb{C}(u)=(\ln(4m+1)-\ln(4m-1))+(\ln(4m-3)-\ln(4m-5))+\ldots>0,$$

consequently

$$0<\mathbb{C}(u)<\ln u$$

(the cases $u\equiv 0,2,3(\operatorname{mod} 4)$ are treated as above). Since $\mathbb{C}(u)=\mathbb{C}([u])$, then for $|s-1|\leqslant\delta$ we have

$$|L(s,\chi_4)|\leqslant(1+\delta)\int_1^{\infty}\frac{\ln u}{u^{2-\sigma}}du\leqslant\frac{1+\delta}{(1-\delta)^2}.$$

Next,

$$L(s,\chi_4)=L(1,\chi_4)+\int_1^s L^{'}(u,\chi_4)du,$$

and

$$|L(s,\chi_4)|\geqslant|L(1,\chi_4)|-|s-1|\frac{1+\delta}{(1-\delta)^2}.$$

Since

$$L(1,\chi_4)=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots=\frac{\pi}{4},$$

we have

$$|L(s,\chi_4)|\geqslant\frac{\pi}{4}-|s-1|\frac{1+\delta}{(1-\delta)^2}.$$

Taking $\delta=\frac{1}{4}$, for $|s-1|\leqslant\frac{1}{\ln x}$, we obtain

$$|L(s,\chi_4)|\geqslant\frac{\pi}{4}-\frac{5}{9}>\frac{1}{5}. \tag{8}$$

Thus, if $\frac{1}{2}\leqslant\operatorname{Re}s\leqslant\frac{5}{8}$, then $|L(2s,\chi_4)|^{-1}\leqslant 5$. In case of $\frac{5}{8}\leqslant\operatorname{Re}s\leqslant 1$ we have

$$|L(2s,\chi_4)|^{-1}\leqslant 1+\sum_{n=2}^{\infty}\frac{1}{n^{2\sigma}}\leqslant 1+\frac{1}{2\sigma-1}\leqslant 1+\frac{1}{2\cdot\frac{5}{8}-1}\leqslant 5.$$

Then

$$F(s)\ll\left(T^{\frac{1-\sigma}{3}}\ln T\right)^2,$$

Consequently,

$$|j_1|\ll\frac{1}{T}\int_{\frac{1}{2}}^{b}T^{\frac{2(1-\sigma)}{3}}(\ln x)^2x^{\sigma}d\sigma=\frac{x}{T}(\ln x)^2\int_{\frac{1}{2}}^{b}\left(\frac{T^{\frac{2}{3}}}{x}\right)^{1-\sigma}d\sigma\ll\frac{x}{T}(\ln x)^2.$$

The same estimation is valid for j_8.

By lemma 4, on the arcs Γ_1, Γ_2 we have

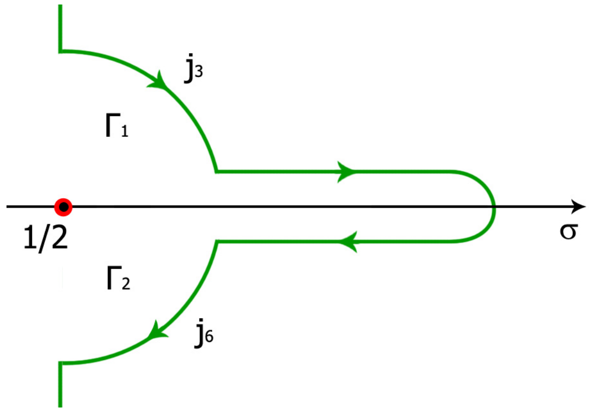
$$|\zeta(s)|\leqslant 3.2,\quad|\zeta(2s)|\leqslant 2.2,\quad|L(s,\chi_4)|\leqslant 3.2,|\zeta(s)|\leqslant 3.2.$$

Using (8), we get

$$|F(s)|\leqslant(3.2)^{\frac{1}{2}}(2.2)^{\frac{17}{48}}5^{\frac{19}{48}}\,|H(s)|<C.$$

Repeating the proofs of the theorems 1 and 2, we get

$$|j_3+j_6|\leqslant\frac{C}{2\pi}\int_{\Gamma_1\cup\Gamma_2}\left|\frac{(x+h)^s-x^s}{s}\right|ds\ll\frac{\sqrt{x}}{\ln x}.$$



Since

$$F(s) \ll |(\zeta(s))|^{\frac{1}{2}} (\ln T)^{\frac{1}{2}}, \quad \text{then}$$

$$|j_2| \ll \left| \int_{\frac{1}{\ln x}}^{T} (\ln x)^{\frac{1}{2}} |\zeta(\frac{1}{2} + it)|^{\frac{1}{2}} \sqrt{x} \frac{dt}{t+1} \right| \ll (\ln x)^{\frac{1}{2}+\frac{5}{4}} \sqrt{x} = (\ln x)^{\frac{7}{4}} \sqrt{x}.$$

The same estimation holds for j_7.

The main term arises from the calculation of j_4 and j_5. As $L(1, \chi_4) = \frac{\pi}{4}$, then, using the proof of the theorem 1, we get:

$$F(s) = \frac{e^{-\frac{\pi i}{4}}}{\sqrt[4]{u}} \Pi(u),$$

where

$$\Pi(u) = H(1-u)(L(1-u, \chi_4))^{\frac{1}{4}} (\zeta(2-2u))^{\frac{17}{48}} (L(2-2u, \chi_4))^{-\frac{19}{48}} \sqrt[4]{w(1-u)}.$$

Suppose that $N \geqslant 0$ is fixed. Then:

$$\Pi(u) = \Pi_0 + \Pi_1 u + \Pi_2 u^2 + \ldots + \Pi_N u^N + O_N(u^{N+1}).$$

Then:

$$j_4 = \frac{1}{2\pi i} \int_{\frac{1}{2}+\frac{1}{\ln x}+i\cdot 0}^{1+i\cdot 0} F(\sigma + i \cdot 0) \frac{(x+h)^s - x^s}{s} ds =$$

$$= \frac{1}{2\pi i} \int_x^{x+h} \int_{\frac{1}{2}+\frac{1}{\ln x}}^{1} F(\sigma + i \cdot 0) y^{\sigma-1} d\sigma \, dy = \frac{e^{-\frac{\pi i}{4}}}{2\pi i} \int_x^{x+h} \int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{\Pi(u) y^{-u}}{\sqrt[4]{u}} du \, dy =$$

$$= \frac{e^{-\frac{\pi i}{4}}}{2\pi i} \int_x^{x+h} \left(\sum_{0 \leqslant n \leqslant N} \Pi_n \int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{u^n y^{-u}}{\sqrt[4]{u}} du + O\left(J\right) \right) dy,$$

where

$$J = \int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{u^{N+1} y^{-u}}{\sqrt[4]{u}} du \ll \frac{\Gamma\left(N + \frac{7}{4}\right)}{(\ln y)^{N+\frac{7}{4}}}.$$

The contribution of the n-th term to the sum is equal to

$$\int_0^{\frac{1}{2}-\frac{1}{\ln x}} \frac{u^n y^{-u}}{\sqrt[4]{u}} du = \frac{\Gamma\left(n + \frac{3}{4}\right)}{(\ln y)^{n+\frac{3}{4}}} + \frac{\theta e n!}{\sqrt{y} \ln y}.$$

Consequently,

$$j_4 = \frac{he^{-\frac{\pi i}{4}}}{2\pi i} \sum_{0\leqslant n\leqslant N} \frac{\Pi_n \Gamma\left(n+\frac{3}{4}\right)}{(\ln x)^{n+\frac{3}{4}}} + O\left(\frac{h}{(\ln x)^{N+\frac{7}{4}}}\right) + O\left(\frac{h^2}{x}\frac{1}{(\ln x)^{N+\frac{7}{4}}}\right),$$

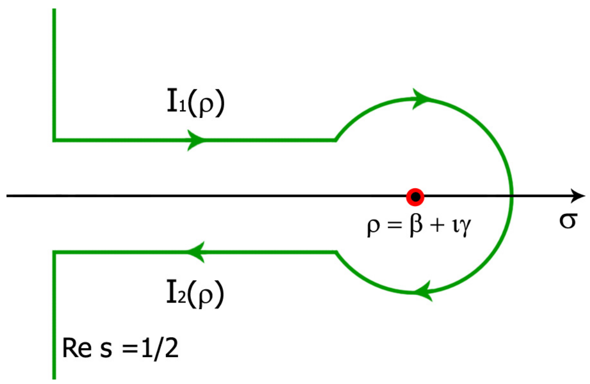
$$j_5 = -\frac{he^{-\frac{\pi i}{4}}}{2\pi i} \sum_{0\leqslant n\leqslant N} \frac{\Pi_n \Gamma\left(n+\frac{3}{4}\right)}{(\ln x)^{n+\frac{3}{4}}} + O\left(\frac{h}{(\ln x)^{N+\frac{7}{4}}}\right) + O\left(\frac{h^2}{x}\frac{1}{(\ln x)^{N+\frac{7}{4}}}\right).$$

By Euler's reflection formula for the Gamma-function,

$$-(j_4+j_5) = -\frac{h}{(\ln x)^{\frac{3}{4}}}\left(\sum_{0\leqslant n\leqslant N} \frac{(-1)^n \Pi_n}{\Gamma\left(\frac{1}{4}-n\right)(\ln x)^n} + O\left(\frac{1}{(\ln x)^{N+1}}\right) + O\left(\frac{h}{x(\ln x)^{N+1}}\right)\right).$$

It remains to estimate the sum

$$\sum_{|\gamma|<T} j_\rho, \quad \text{where} \quad \rho = \beta + i\gamma, \;\; j_\rho = I_1(\rho) + I_2(\rho).$$



Firstly, consider the sum over the zeroes of $\zeta(s)$. Using the same arguments as above, we get

$$I_1(\rho) \ll \frac{h}{x}(\ln x)^{\frac{1}{2}} \int_{\frac{1}{2}}^{\beta} x^\sigma |\zeta(\sigma+i\gamma)L(\sigma+i\gamma,\chi_4)|^{\frac{1}{4}} d\sigma.$$

Therefore,

$$|j_\rho| \ll h\ln x \int_{\frac{1}{2}}^{1} g(\rho,\sigma)\left(\frac{T^{\frac{1}{6}}}{x}\right)^{1-\sigma} d\sigma,$$

where

$$g(\rho,\sigma) = \begin{cases} 1, \text{ if } \sigma \leqslant \beta, \\ 0, \text{ if } \sigma > \beta. \end{cases} \tag{9}$$

Applying the lemma 6,

$$\sum_{|\gamma|<T} j_\rho \ll h(\ln x)^{45}\left(\frac{T^{\frac{12}{5}+\frac{1}{6}}}{x}\right)^{1-\varrho(T)}.$$

Now let us evaluate the sum over the zeroes of $L(s,\chi_4)$. We have:

$$I_1(\rho) \ll h(\ln x)^{\frac{3}{4}+\frac{17}{48}} \int_{\frac{1}{2}}^{\beta} x^{\sigma-1} T^{\frac{1-\sigma}{6}} d\sigma \ll h(\ln x)^{\frac{53}{48}} \int_{\frac{1}{2}}^{1} g(\rho;\sigma) \left(\frac{T^{\frac{1}{6}}}{x}\right)^{1-\sigma} d\sigma,$$

where $g(\rho;\sigma)$ is defined by (9). Applying lemma 6, we get:

$$\sum_{|\gamma|<T} j_\rho \ll h(\ln x)^{\frac{41}{48}} \int_{\frac{1}{2}}^{1-\varrho(T)} N(\sigma;T,\chi_4) \left(\frac{T^{\frac{1}{12}}}{x}\right)^{1-\sigma} d\sigma \ll$$

$$\ll h(\ln x)^{\frac{53}{48}} \int_{\frac{1}{2}}^{1-\varrho(T)} \sum_{\chi \bmod 4} N(\sigma;T,\chi_4) \left(\frac{T^{\frac{1}{6}}}{x}\right)^{1-\sigma} d\sigma \ll$$

$$\ll h(\ln x)^{45+\frac{53}{48}} \int_{\frac{1}{2}}^{1-\varrho(T)} \left(\frac{T^{\frac{1}{6}+\frac{12}{5}}}{x}\right)^{1-\sigma} d\sigma.$$

Choosing T from the equation

$$T^{\frac{12}{5}+\frac{1}{6}} = xD^{-1}(x),\ D(x) = e^{C_1(\ln x)^{0.8}}$$

we get

$$T = x^{\frac{30}{77}} D(x)^{-\frac{30}{77}}.$$

Now we conclude that the formula

$$j_0 = \frac{h}{(\ln x)^{\frac{3}{4}}} \sum_{0\leqslant n\leqslant N} \frac{(-1)^n \Pi_n}{\Gamma(\frac{1}{4}-n)(\ln x)^n} + O(J),$$

$$J \ll \frac{x}{T}(\ln x)^{\frac{17}{4}} + \frac{h}{(\ln x)^{N+\frac{7}{4}}} + h(\ln x)^{47} \left(\frac{T^{\frac{12}{5}+\frac{1}{6}}}{x}\right)^{1-\varrho(T)},$$

is asymptotic, if

$$h = x^{\alpha} e^{C_2(\ln x)^{0.8}} \gg \frac{x}{T}(\ln x)^2,$$

where

$$\alpha = 1 - \frac{30}{77} = \frac{47}{77},$$

which proves the theorem.

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